N Compartments with Delay: Transit Time

The ordinary differential equation for a one compartment model with flow assumed to be constant is given as:

$$Volume \cdot dC(t)/dt = Flow \cdot Cin(t) - Flow \cdot Cout(t).$$

Since the compartment is instantaneously well mixed, this equation can be written as

$$dC/dt = (F/V) \cdot (Cin - C).$$

(We have dropped "(t)", replaced Cout with C, and divided both sides by V after shortening Volume to V and Flow to F.

This is an ordinary differential equation (ODE) and requires an initial condition for solution. Choose

$$C(0) = C0,$$

which means at time equals zero, the concentration is C0 (some value). Choose Cin(t), the inflowing concentration of material to be zero. The solution is

$$C(t) = C0 \cdot \exp(-F \cdot t/V).$$

Substituting $\tau = V/F$, we have

$$C(t) = C0 \cdot \exp(-t/\tau),$$

where τ has the units of time, so that the exponential operates on a dimensionless quantity.

4. AREA, TRANSIT TIME $\bar{t}$, VARIANCE (SD2), and RELATIVE DISPERSION (RD)

The area of an outflow curve is calculated as

$$Area = \int_0^\infty C(t)\,dt \quad .$$

For $C(t) = C0 \cdot \exp(-t/\tau)$, $Area = \tau \cdot C0.$

The transit time of an outflow curve is calculated as

$$\bar{t} = \frac{\int_0^\infty C(t) \cdot t\,dt}{Area.} \quad .$$

For $C(t) = C0 \cdot \exp(-t/\tau)$, $\bar{t} = tau.$

The variance of an outflow curve is calculated as

$$SD^2 = \frac{\int_0^\infty C(t) \cdot (t - \bar{t})^2\,dt}{Area.} \quad .$$

For $C(t) = C0 \cdot \exp(-t/\tau)$, $SD^2 = \tau^2.$

The relative dispersion of an outflow curve is calculated as

$$RD = \frac{\sqrt{SD^2}}{\bar{t}}.$$

For $C(t) = C0 \cdot \exp(-t/\tau)$, $RD = 1$.

Now,

$$\overline{t_{input}} + \overline{t_{system}} = \overline{t_{output}}.$$

The transit time of an input curve plus the transit time of the "system" equals the transit time of the output curve. If there is no consumption or synthesis in an arrangement of compartment models, the transit time is also given by

$$\bar{t} = \overline{t_{system}} = \frac{\sum_{i=1}^{n} V_i}{Flow},$$

where V_i are the connected volumes. Variances also add:

$$SD^2_{input} + SD^2_{system} = SD^2_{output}.$$

These two equations are true as long as everything that enters the "system" as input exits the system as output. From the input and output curves (inflow and outflow concentrations), the transit time and the relative dispersion of the system can be determined. The transit time and relative dispersion for a compartmental systems are independent of the shape of the input and output functions as long as everything that enters the system also exits the system.

CONSEQUENCES:

What is the RD of N-stirred tanks in series, each with volume V/N?

Each tank has transit time

$$t_{SINGLE\,TANK} = \frac{(V/N)}{Flow} = \tau / N.$$

The transit time of the entire system is

$$t_{SYSTEM} = N \cdot t_{SINGLE\,TANK} = \tau.$$

Each tank has variance

$$SD^2_{SINGLE\,TANK} = \left(\frac{tau}{N}\right)^2.$$

The total variance of the system is

$$SD^2_{SYSTEM} = N \cdot SD^2_{SINGLE\,TANK} = \left(\frac{tau^2}{N}\right).$$

The relative dispersion of the system is

$$RD_{SYSTEM} = \frac{\sqrt{SD^2_{SYSTEM}}}{t_{SYSTEM}} = \sqrt{\frac{1}{N}}.$$

The relative dispersion in arteries of the human leg has been determined to be 0.18 (1) which would require 31 stirred tanks in series (*RD = 0.1796*).

What is the RD of N-stirred tanks in series, each with volume V/N with an added delay?

Each tank has transit time

$$t_{SINGLE\,TANK} = \frac{(V/N)}{Flow} = \tau / N.$$

The transit time of the entire system is

$$t_{SYSTEM} = N \cdot t_{SINGLE\,TANK} + delay = \tau + delay.$$

Each tank has variance

$$SD^2_{SINGLE\,TANK} = \left(\frac{tau}{N}\right)^2.$$

The total variance of the system is

$$SD^2_{SYSTEM} = N \cdot SD^2_{SINGLE\,TANK} = \left(\frac{tau^2}{N}\right).$$

The relative dispersion of the system is

$$RD_{SYSTEM} = \frac{\sqrt{SD^2_{SYSTEM}}}{t_{SYSTEM}} = \frac{1}{\sqrt{(N)} * (1 + delay/\tau)}.$$

References:

1. Bassingthwaighte, J.B. Plasma indicator dispersion in arteries of the human leg. Circ. Res. 19: 332-346, 1966.