Appendix B is excerpted from:

Bassingthwaighte, J. B. and D. A. Beard. Fractal ^{15}O-water washout from the heart. *Circ. Res.* 77:1212-1221. 1995.

B: Sums of Scaled Functions Can Give Power Law Behavior

A power law function can be represented as the sum of a finite number of fractal-scaled basis functions. Consider approximating of the power law function of Equation 16 with the weighted sum of basis functions *f*(*t*) in Equation 17:

$$F = t^{-\beta} \tag{16}$$

$$F \approx \sum_{i=1}^{N} a_i \mathrm{f}(k_i t) \tag{17}$$

where a_i is the amplitude scalar and k_i is the time scalar for the ith member. Since the basis functions are not necessarily orthogonal, a finite sum of *N* scaled basis function is considered.

The minimum mean-squared error between *F*(*t*) and a particular $f(k_i t)$ over the interval from t=0 to t=∞ is found by calculating a_i:

$$a_i = \frac{\int_0^\infty f(k_i t) t^{-\beta}\ dt}{\int_0^\infty f^2(k_i t)\ dt} \tag{18}$$

From this, one can solve the relation between a_i and k_i by using a dummy variable, $\tau = k_i t$, substituted into Equation 18:

$$a_i = \frac{k_i^{\beta-1}\ \int_0^\infty f(\tau)\tau^{-\beta}\ d\tau}{\frac{1}{k_i}\ \int_0^\infty f^2(\tau)\ d\tau} = \left(\frac{\int_0^\infty f(\tau)\tau^{-\beta}\ d\tau}{\int_0^\infty f^2(\tau)\ d\tau} k_i^{\beta} \right. \tag{19}$$

or

$$a_i = C k_i^{\beta} \tag{20}$$

where *C* is a constant that does not depend on k_i.

A power law function can therefore be represented by a finite sum of the scaled basis functions, where the weight of each basis function is determined by the scale factor raised to the power law exponent:

$$F \approx C \sum_{i=1}^{N} k_i^{\beta} f(k_i t) \tag{21}$$

In general, the k_i can be chosen on the basis of the interval over which the power law slope is fit. If the interval is defined by $t=t_a$ to $t=t_b$, then k_1 can be chosen by $k_1=1/t_a$ or a conveniently chosen value. In order to evenly distribute all of the k_i in the log-time domain, the rest of the k_i can be calculated over the range chosen:

$$k_i = \left(\frac{k_N}{k_1}\right)^{\frac{i}{N-1}} = \left(\frac{t_a}{t_b}\right)^{\frac{i-1}{N-1}} k_1 \tag{22}$$

An example using exponentials as the basis function is demonstrated in Fig 7. *F* and *f* are given as follows:

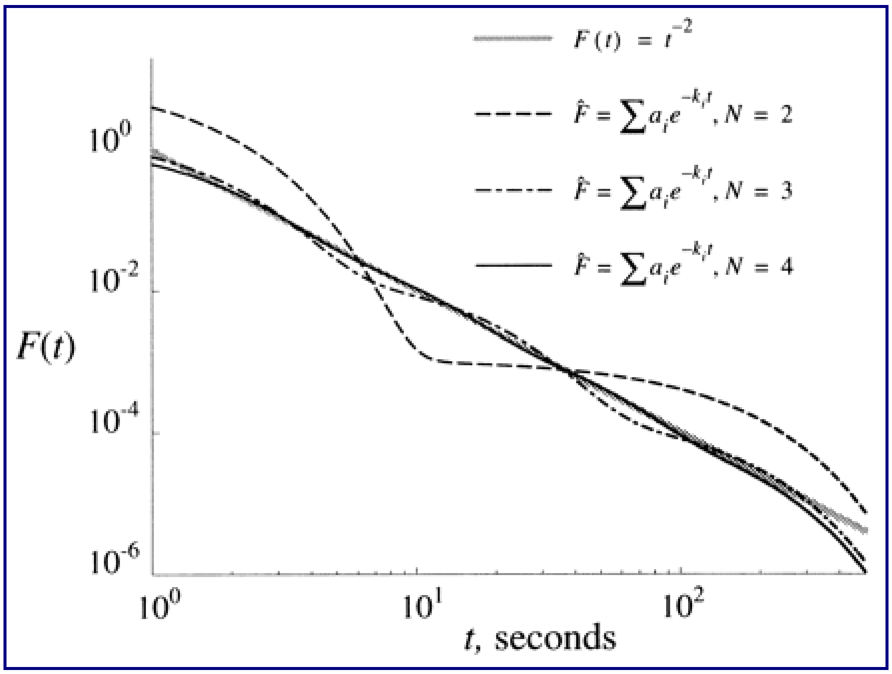

Figure 7. Log-log plot showing multiexponential fits to $F=t^{-2}$, where *F* is flow and *t* is transit time, using two, three, and four exponentials with $t_a=1$, $t_b=100$, $k_2/k_1=100$ for $N=2$, $k_2/k_1=10$ for $N=3$, and $k_2/k_1=4.642$ for $N=4$, where *k* is the time scalar, and *N* is the number of exponentials.

$$F = t^{-2} \tag{23}$$

$$f(t) = e^{-t} \tag{24}$$

The finite-sum approximation is shown for N=2, 3, and 4 exponentials. An approximate fit is achieved using only four exponentials over the interval of t_a=1 to t_b=100. Making t_a and t_b outside of the desired region to be fitted and increasing N allows one to approach exact power law behavior arbitrarily closely.