Chapter 14

Convective Transport of Fluids

14-1. Fluid displacement: flow through a pipe

There are many representations of flow through a pipe. The one most commonly used is the one developed independently by Poiseuille (1846) and Hagen (1841). It was based on observations of the steady flow of water and other solutions in smooth straight cylindrical tubes. From the data on the observed flows, the pressures at inflow and outflow, viscosities, tubing length and radius, Poiseuille derived the summarizing equation,

$$F = \frac{\Delta P R^4}{8\pi\eta L}, \tag{14-1}$$

where ΔP is the pressure difference between inflow and outflow, R is radius, η is viscosity of the fluid, and L is the length of the tube. This is just Ohm's law, with F = current and P = voltage; the resistance is $8\pi\eta L/R^4$. (The units are given in the Terminology in Chapter 1.) The beauty of this simple expression, either Poiseuille's or the more general Ohm's Law version, Flow = Pressure/Resistance, is that it is not only an excellent quantitative descriptor over a wide range of conditions but in addition describes qualitatively the relationships under a yet broader range of conditions even where Poiseuille's conditions do not hold: higher pressure causes more flow, higher viscosity decreases flow, greater length gives greater resistance, and larger radii give reduced resistance and higher flow. All of these are intuitively obvious to us nowadays. But where did the 8 come from? The theory is in the next chapter.

Now change the view point from that of assessing flow/pressure relationships to assessing input/output relationships. This I/O view going to be brought back to mesh with that of pressure/flow studies, but it focuses on observations of the fate of fluid molecules entering the system, not on mechanism. Questions are: What is the transit time from entrance to exit? How is a specific concentration-time curve for a solute at the entrance deformed in the processes undergone during that transit? The ideas are based on Conservation of Mass coupled with the effects of diffusion or other sources of dispersion. Tracer principles come into play in such experiments; radioactive tracers can be detected by sampling the flowing fluid at which frequency to characterize the input and output concentration-time curves. Moreover, gamma emitting tracers can be detected while inside the system by placing gamma detecting NaI(Tl) crystals (sodium iodide, thallium activated) externally. (Such external detection is the basis of noninvasive imaging procedures.)

This chapter takes a "model-free" approach. (Never trust anyone who says a "model-free" approach is used. There is always an underlying set of assumptions which comprise, of course, a model. Figure out how to make such assumptions explicit, and so reveal the underlying hypotheses.) In our "model-free" approach we assume certain fundamentals: (1) no mass loss between inflow and outflow, (2) flow is steady, and (3) there is no binding of tracer or fluid molecules to the wall of the vessel. Then more assumptions will be made, all explicitly.

14-1.1. Simple fluid displacement without dispersion

Consider the lowly pipe. Unless it is clogged or leaks, what comes in goes out. This is mass balance—nothing lost, nothing gained. Figure 11-11 shows an idealized frictionless pipe, one in which the fluid does not stick to the walls. Sticking only to itself (internal viscosity) but not to walls allows a flat velocity profile.This is the explicit model *assumption, fourth* one on top of the three listed above. (A velocity profile is defined as the map of velocities at each radial position across a cross section of the pipe, the relation v(r); in this particular case v(r) = constant.) Now a *fifth assumption*: there is no molecular diffusion spreading the fluid particles upstream or downstream; they all move with the same velocity. When the velocity profile is flat, disc-like within the tube, and there is no dispersion axially, it doesn't matter if there is or is not fluid rotation, since this will not influence the arrival time of the fluid particles at the outflow.

14-1.2. A special input, the impulse (so short that it happens only in theory).

A flat or uniform velocity profile in the pipe (Fig. 11-11) makes description of the mass balance simple. The output is merely a delayed replica of the input. When the input to such a mathematically nice pipe is a very short pulse, close to the idealized Dirac delta function, $\delta(t)$, then the uninteresting result is a spike in, and later a spike out (Fig. 11-12). What is useful is a measure of the delay between input and output.

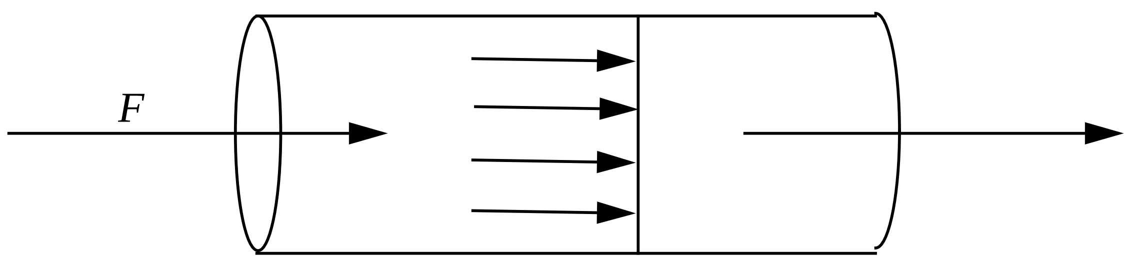

Figure 14-1: Fluid displacement without dispersion, a uniform velocity profile, often called piston flow.

Consider the line in Fig. 11-11 to designate the front of a column of fluid, labeled just at the interface between the column of fluid filling the pipe and the new fluid which is about to displace it. Now the new fluid advances at a constant velocity, the same at all points across the cross section, pushing out the old. When the labeled front of the piston emerges, the volume of new fluid has just sufficed to push out the old. The time, t_0, required to move the column of new fluid, the piston, from the entrance to the exit is calculated exactly by

$$t_0 = V_{\text{pipe}} / F , \tag{14-2}$$

where V_{pipe} is the volume of the pipe and F is the flow (volume/unit time). The calculated t_0 and the observed delay must match. The delay is $t_{\text{out}} - t_{\text{in}}$, where t_{in} is the appearance time of the input concentration and t_{out} the appearance time of the output concentration. This must equal t_0 if the flow is steady and no fluid is lost (conservation of mass). The mean transit time, $\bar{t}$, through the pipe is t_0, all particles having this same transit time.

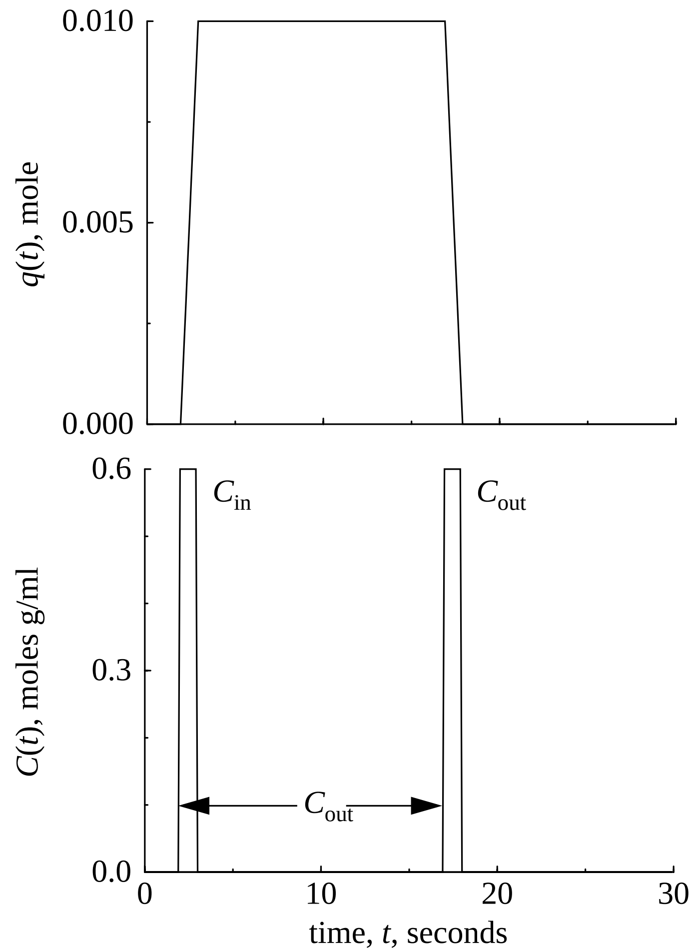

Figure 14-2: Brief pulse input of 10 millimoles in 1.0 second into a pipe. C_{in} is the concentration at the inflow, a pulse beginning at $t = 2$ s and ending at $t = 3$ s, and C_{out} is that at the outflow end of the pipe, a pulse beginning at $t = 17$ s and ending at 18 s. The delay, t_0, is the transit time. The upper panel records the amount of material in the pipe, as if by external detection. $F = 0.0167$ ml g^{-1} s^{-1} = 1 ml g^{-1} min^{-1}, $V = 0.25$ ml g^{-1}, and $t = V/F = 15$ s.

The amount of the tracer label within the pipe is graphed in the upper panel of Fig. 11-12. It is the signal that would be obtained by external detection of a gamma-emitting or positron-emitting tracer, using a detector that recorded emission coming only from the tracer inside the pipe. This is the residue function—the sum of all the tracer that has entered the pipe but not yet left the exit. In this idealized example, the residue function rises from 0 to 0.01 moles between $t = 2$ seconds and $t = 3$ seconds for the whole of the tracer entered during that time. The residue function declines from 0.01 to 0 moles between $t = 17$ seconds and $t = 18$ seconds. Between $t = 3$ seconds and $t = 17$ seconds all of the tracer was inside the pipe, was seen by the detector, and gave a constant signal.

In a formal sense, when the input is an infinitely short pulse and yet introduces a unit mass, $C_{in}(t) = \delta(t)$, then $C_{out}(t) = \delta(t - t_0)$. The Dirac delta function or impulse input $\delta(t)$ is formally defined as an infinitely narrow, but infinitely high spike occurring at $t = 0$, and having unity area.

At times before and after $t = 0$, its value is zero. Accordingly $\delta(t - t_0)$ is a spike of unit area occurring at time $t = 0$ and having zero values before and after. One can think of it as a unit area Gaussian normal curve with the standard deviation reduced to zero. For an input of any form, $C_{in}(t)$, when the impulse response is a delayed Dirac delta function, undispersed, then the output, $C_{out}(t)$, has the same value as the input, but is delayed by t_0:

$$C_{out}(t) = C_{in}(t - t_0) \ , \tag{14-3}$$

or in words, that the output is identical to the input except for a delay of t_0. When material is entering, then the residue function $R(t)$ accumulates all that enters, i.e., it is the integral. Formally, this is represented in the special case where the input is a delta function, $\delta(t)$, as a step function; at time 0 the contents of the pipe increased by the unit step, and at time t_0 the contents decreased by a unit step, $S(t)$:

$$q(t) = S(t) - S(t - t_0) \ . \tag{14-4}$$

In this case the unit step function is appropriate because exactly one unit of material entered at time 0 and emerged at t_0. The area under the curve of $q(t)$ is the transit time itself in this case because one unit of tracer was used. In general, the area of $q(t)$ divided by the amount entering, q_0, gives the transit time.

14-1.3. A clean but broad pulse input

Injecting a broader bolus with the same total amount of tracer but over several seconds rather than as a short impulse gives the result shown in Fig. 11-13. The relationship between C_{in} and C_{out} is the same, they are identical in shape and C_{out} is delayed by t_0. The content of the pipe, $q(t)$ (upper panel), is changed in form: the longer pulse entering results in a linear increase in content until all has entered, but as before the plateau is steady until the first tracer exits, then has a linear decrease. These ramps, in and out, illustrate another generality: just as the step function is the integral of the impulse, the ramp function is the integral of the step. The ramp increase will continue so long as the input is a constant, or, for a continuous inflow, until outflow begins. With the longer input pulse, the plateau shortens.

14-1.4. A more general dispersed input

When C_{in} is dispersed in some nondescript way, the same principles apply, as shown in Fig. 11-14. The output is just as precisely defined as before, i.e., $C_{out}(t) = C_{in}(t - t_0)$. The shape of the residual content of the pipe is predictable from the input, for it integrates what enters until there is outflow, and then subtracts the integrated outflow:

$$q(t) = F\int_0^t (C_{in}(\lambda) - C_{out}(\lambda))d\lambda \ . \tag{14-5}$$

If one were to collect all the outflow in a beaker, one would gather the total dose, q_0:

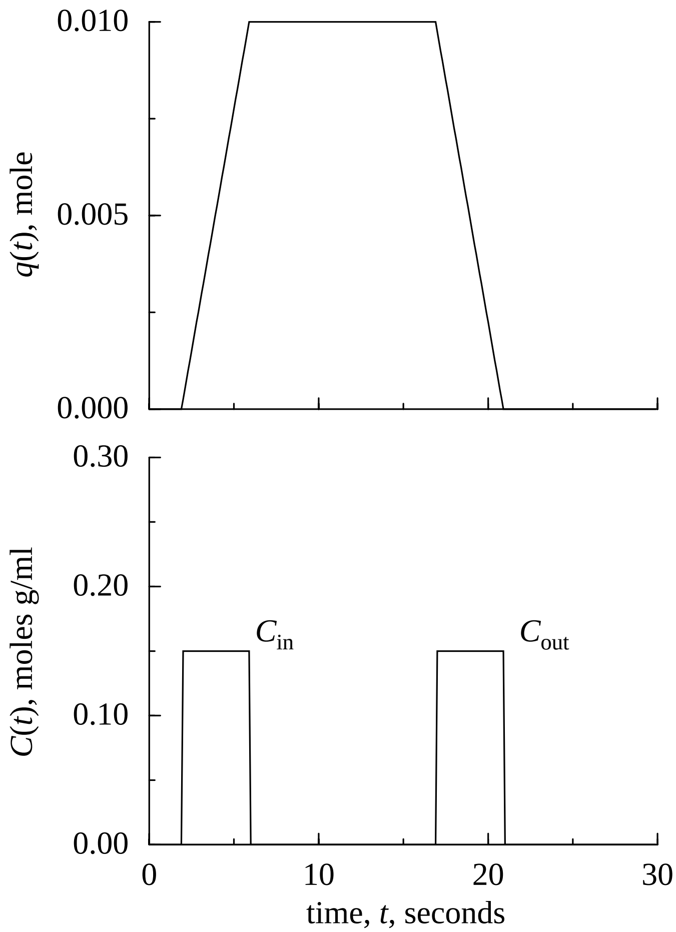

Figure 14-3: Input and output concentration-time curves for a dispersionless pipe with finite pulse input. Same F and V as in Fig. 11-12. The total injected, $q_0 = 10$ mmole is also the same, but the input pulse is 4.0 seconds, four times as long as that in Fig. 11-12.

$$q_0 = F\int_0^{\infty} C_{\text{in}}(t)dt = F\int_0^{\infty} C_{\text{out}}(t)dt \quad . \tag{14-6}$$

14-1.5. A small pipe begins to lose material before the input is complete

Figure 11-15 illustrates what is commonly found when one examines the content of a Region Of Interest (ROI) in a PET (Positron Emission Tomography) signal, particularly when the region is small. Material starts to leave so soon after entering that not all of the injected material has entered. When this happens the plateau is never reached. (This is compromising to analyses that are based on measuring the total amount of tracer that entered a region. Using a deposited indicator, regional flow is estimated from the tracer content of the ROI, but when tracer leaves before the entry is complete the contents of various ROIs at any particular moment are not proportional to the local flows.) Figure 11-15 shows a family of outflow and content curves for various pipe volumes. If the input, C_{in}, is brief enough, then even a small volume pipe would contain the complete bolus of tracer. The practical problem is to achieve a short input function;

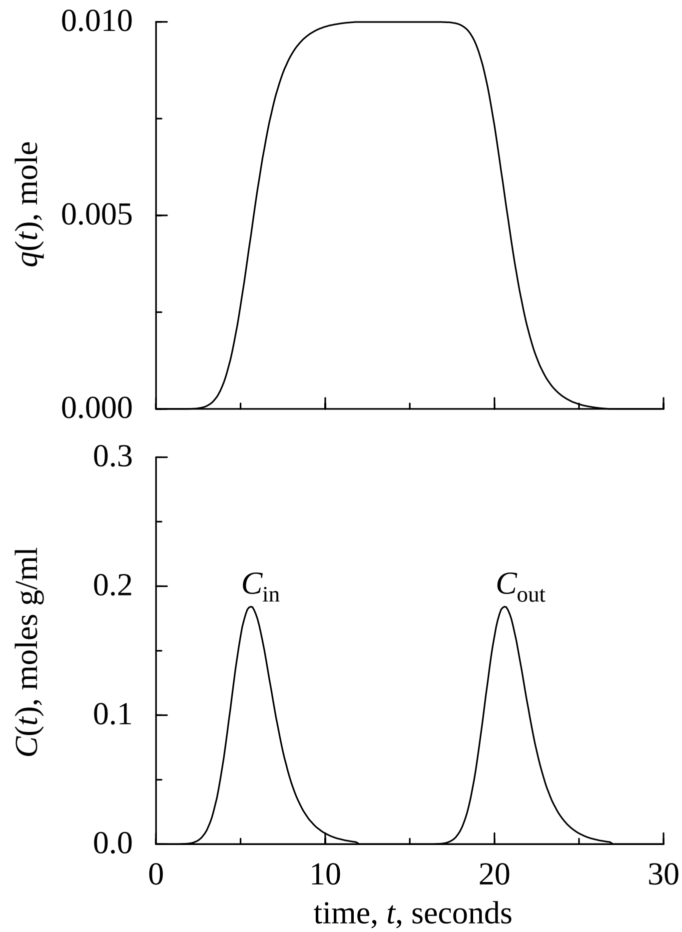

Figure 14-4: Responses of the same system as in Fig. 11-13 to a dispersed input. (Input is a lagged normal density curve [Bassingthwaighte, Ackerman, and Wood, 1966] with $\sigma = 0.96$, $\tau = 1.15$ seconds, and $t_C = 4.85$ seconds.)

one needs to make injections short and close to the entrance into the organ or pipe. All vascular transport is dispersive, that goes for large arteries and veins, as well as in capillaries, making it impossible for an injection to reach a small region as a nice square pulse.

14-2. Fluid displacement with dispersion

Dispersionless piston-flow is unattainable in reality. A thin cross-sectional lamina of fluid is soon spread out along a tube. The mechanisms vary in importance in different flow regimes and differently structured tubes, but they include: (1) variation in velocities at different points in the cross section, the velocity profile; (2) molecular diffusion axially and radially, prominent at very low flows; (3) turbulent dispersion, which is random like molecular diffusion; (4) eddies at branches and bends; (5) pulsatile flow; (6) the particulate nature of blood, with red cells and plasma being separated near walls, particularly in small vessels; and (7) fluid viscosity changing with shear rate (thixotropy) and hematocrit.

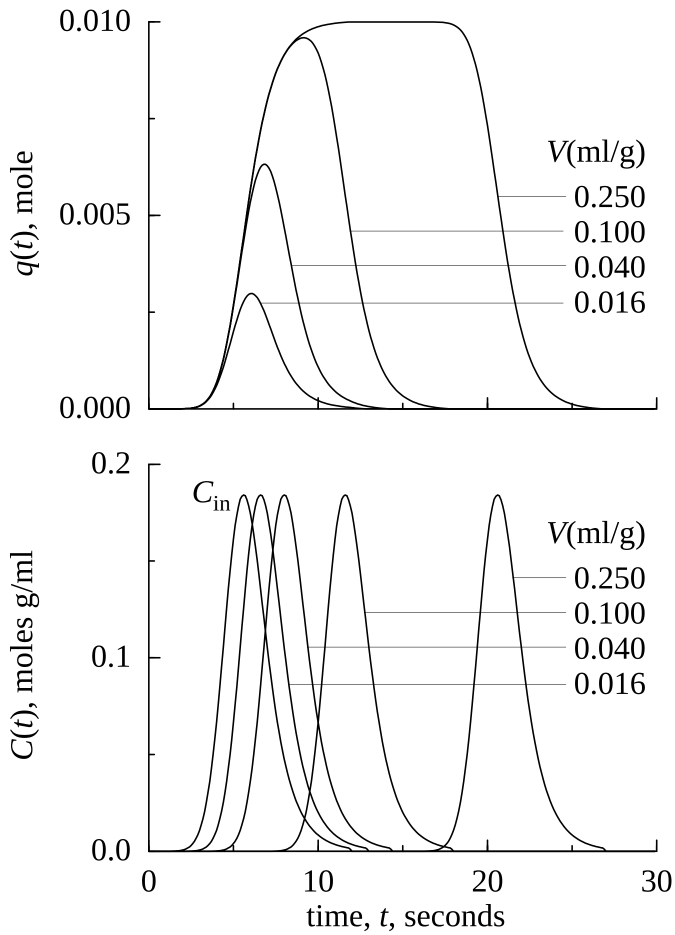

Figure 14-5: Effect of diminishing volume on the system responses to a dispersed input (lower panel). The system content fails to achieve a plateau when *V* is small (upper panel).

14-2.1. Comparing dispersionless and dispersive systems

The same system responses are shown in Fig. 11-16 for a brief pulse injection into a pipe where dispersion is allowed. Now we begin to approach reality, because in pipes in which there is friction between fluid and wall the velocities are lower at the wall, higher in the center of the stream, so tracer in the central laminae of the flow streamlines and will arrive sooner at the outflow. This is dispersion by the velocity profile. When dispersion occurs because of turbulence or disturbed flow streams, at high or low flows, or by molecular diffusion that randomizes the position of the tracer molecules, then the end result is similar: tracer entering as a narrow bolus is spread in space and time of arrival at the outflow. The two examples shown in Fig. 11-16 illustrate the role of dispersion in complicating the form of the residual tracer content. When the volume of the pipe is large enough, the dispersion has no influence on the plateau in pipe content, $q(t)$. For smaller volumes, the plateau is achieved because the input waveform is a rather brief clean pulse, but is not maintained very long because the dispersion within the pipe leads to earlier loss through the outflow. If the pipe had a significantly smaller volume, the plateau would not be reached.

In Fig. 11-17 the contrast between a brief pulse input and a dispersed input is more evident. The upper panel shows the effect of diminishing the volume on the retained tracer content of the

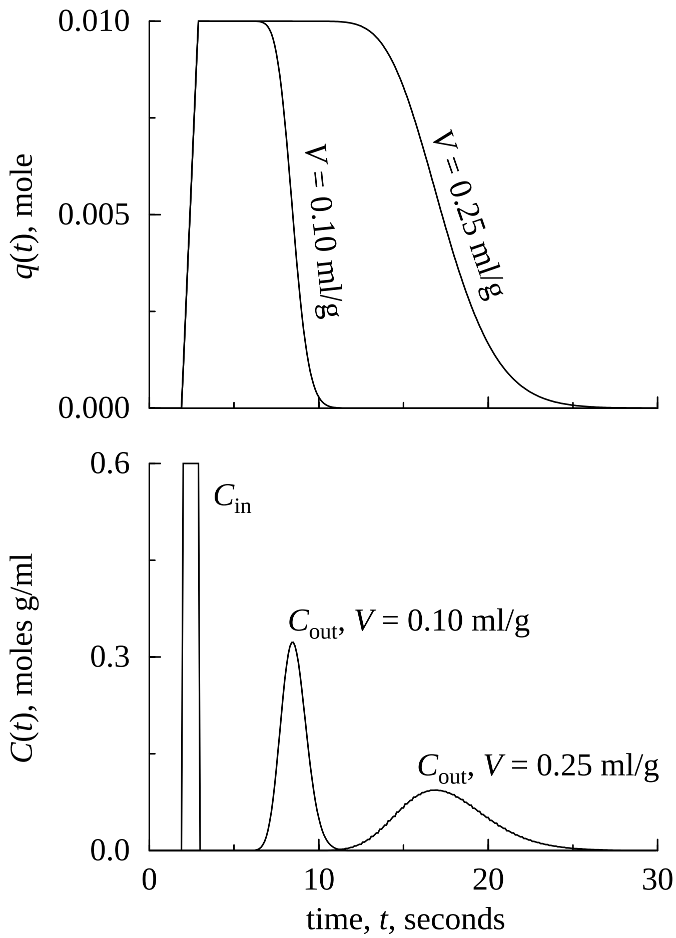

Figure 14-6: Responses to a pulse input in a system with dispersion. When the volume is sufficiently large relative to the volume of fluid containing the input, a plateau is reached even when there is internal dispersion. However, internal dispersion abbreviates the plateau, and may even reduce the maximum tracer content when the volume is smaller. $F = 1$ ml g^{-1} min^{-1}.

pipe, $q(t)$, when the input is brief. The plateau is reached even for the smaller volumes. The lower panel shows that the plateau is not achieved with smaller volumes when the input is more dispersed.

14-3. A formal notation

The notation defined originally by Zierler (Meier and Zierler, 1954; Zierler, 1962 and 1965) and formalized by an international agreement (Bassingthwaighte et al., 1986), summarizes what has been represented for transport through an idealized pipe. When the input is the Dirac function, $\delta(t)$, then the system responses are given formal names; these names are completely general. When any other input occurs, the system responses can be defined from the formal descriptors by minor arithmetic manipulations, usually a convolution. See Fig. 11-18.

The impulse input occurs at $t = 0$. The system response to it at the outflow is the transport function $h(t)$, defined by Zierler as the frequency function of transit times, or the probability

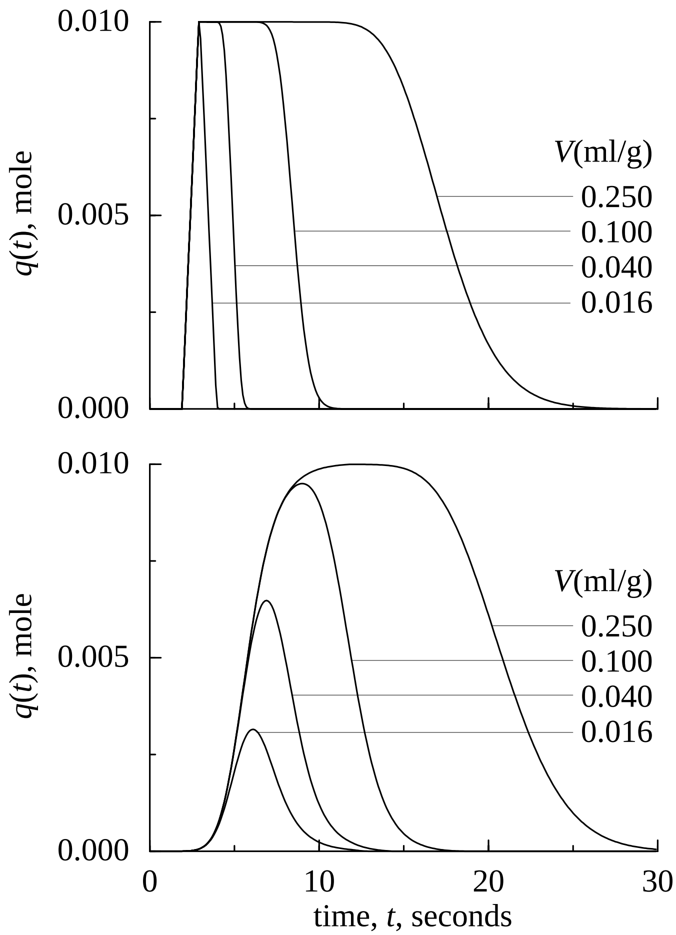

Figure 14-7: Tracer content of a dispersive system at various volumes when the input is a pulse of 1.0 second duration (upper panel), or a dispersed input form (lower panel). With a small system volume, the peak, $q(t)$, reaches the plateau representing the total dose only when the input concentration-time curve is shorter than the shortest transit time.

density function of transit times, through the system. The accumulation of all the material flowing out, as if in a bucket at the exit, is the residence time distribution function, $H(t)$. In this special case, everything has entered the system at $t = 0$, so that the residue function $R(t)$ is 1.0 at $t = 0$, and remains at that level until tracer begins to exit. The residue function is the complement of what has emerged:

$$R(t) = 1.0 - H(t) . \tag{14-7}$$

The rate at which material emerges is $h(t)$, the fraction of the material injected as an impulse which emerges per unit time. Another way of looking at the system is from the point of view of the contents. The rate of exit $h(t)$ is the rate of diminution of $R(t)$:

$$dR(t)/dt = -h(t) . \tag{14-8}$$

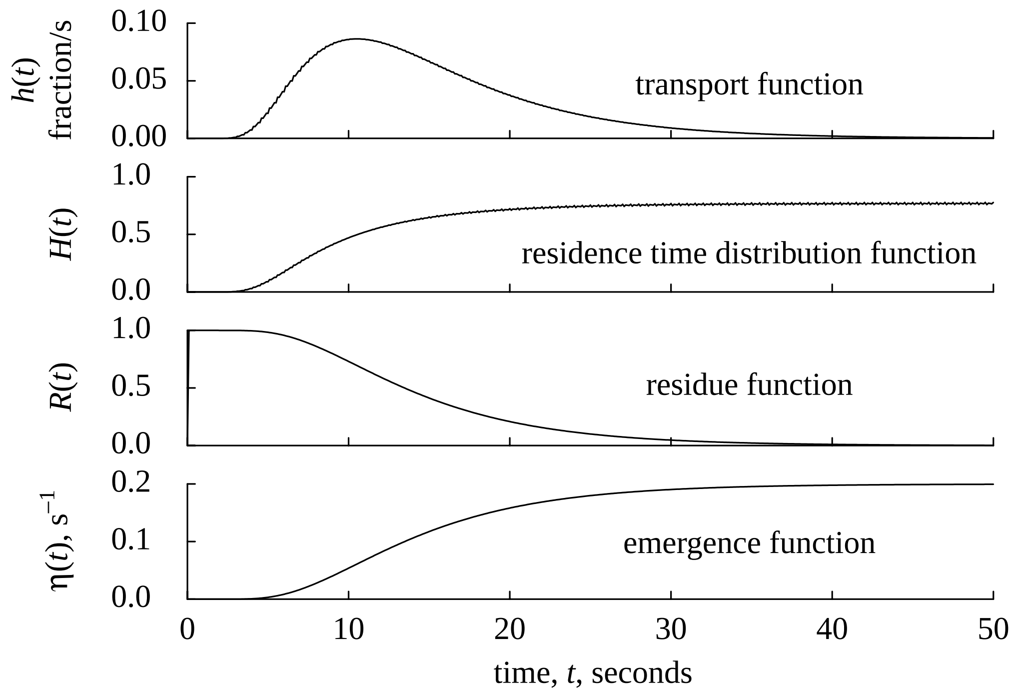

Figure 14-8: Mass transport through a stationary system: relationships between $h(t)$, $H(t)$, $R(t)$, and $\eta(t)$. Input is an approximation to $\delta(t)$, namely a 0.1 s duration pulse starting at $t = 0$.

Again from the point of view of the contents, the fractional rate of loss is the rate of exit *h(t)* divided by the contents itself. This defines the *fractional escape rate for an impulse response*, the emergence function, $\eta(t)$ is

$$\eta(t) = h(t)/R(t) \quad . \tag{14-9}$$

This is the negative of the slope of the residue function, which we see most commonly as a washout curve recorded by external detection.

14-4. Illustrative Program: *Simpipe*

Simpipe is a SIMCON program providing an input subroutine, *cinput*, and an option to choose one of two delay-type operators, btex10 (*bt10*) or *dlymn*, to represent the pipe.

Everything in this section can be reproduced with a single operator, *bt10*, which is an axially distributed blood–tissue exchange model that acts as a nondispersing delay when the dispersion coefficient is set to 0, otherwise it acts as a dispersive delay. Because the coding of *bt10* provides for a maximum delay of 60 intervals of its internal time step (may be longer or shorter than the user-chosen time step, when the transit time is very long one may prefer to use a pure delay line when the dispersion is zero, so a second delay operator (*dlymn*) is provided and is governed by the same parameters *F* and *V*.

The input function is generated by *cinput*, a program allowing a large variety of input wave forms. See UNIX manual pages under linux on the NSR system, for cinput, btex10, and dlymn.

14-4.1. XSIM parameter arrays for figures using simpipe:

To run the programs for each of the figures in this chapter over the web go to
http://nsr.bioeng. washington.edu
Click on Software, then on Blood-tissue exchange models, then on simpipe, and then choose the particular figure you want by clicking on the relevant parameter file, e.g., simpipe2.par for Fig. 11-12. When the XSIM control window comes up, click on Results, then Plot Area 1 and then move the plot window down to the lower right part of the screen so you can access the control window.

Click Run for a solution. To display other variables on the graph right click in a variable box (left upper) and choose a variable from the list that comes up. You may have to change the scaling for the Y variable (right upper part of Plot Area).

14-4.2. Programs archived and available that relate to this chapter:

These are available for those using Unix or Linux systems:

- Subroutines—*cinput, bt10, dlymn*, and their related subroutines.
- Programs to produce illustrations—*simpipe.*

14-4.2.1. XSIM parameter arrays for figures using *simpipe*

These are available at http://nsr.bioeng. washington.edu:

- For Figure 2—*simpipe2.par*
- For Figure 3—*simpipe3.par*
- For Figure 4—*simpipe4.par*
- For Figure 5—*simpipe5.par*
- For Figure 6 —*simpipe6.par*
- For Figure 7 (top and bottom)—*simpipe7T.par, simpipe7B.par*
- For Figure 8—*simpipe8.par*

14-4.2.2. JSIM program and parameter sets (XSIM)

The JSIM program and parameter sets (XSIM) for comparable figures can be found at /user12/garyr/public_html/BTEX/BTEXPLUS. Copy btex10plus_pde.mod and the parameter sets to your own directory.

14-5. Problems:

14-6. Chapter Summary:

14-7. Further readings:

Bassingthwaighte and Goresky (1984) give a general description of the phenomenology and of these formal descriptors. The terminology we use here follows that proposed by an international committee (Bassingthwaighte et al., 1986). Stephenson (1948) gave the general theory in a formal mathematical fashion, but the several papers of Zierler are more readily digestible and are excellent sources from which to learn. Methods of applying residue function analysis are described by Bassingthwaighte (1977) and Bassingthwaighte et al. (1993).

14-8. References

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