The Lagged Normal Density Function

As a starting point for the descriptions of the transport function (the probability density of transit times through a flowing system), the lagged normal density function (LAGNDC) is by no means the only one to choose but it works well for large arteries (Bassingthwaighte, Ackerman, and Wood, 1966; Bassingthwaighte, 1966), even the aorta (Bassingthwaighte and Ackerman, 1967) and the lung (Knopp and Bassingthwaighte, 1969), and as a component in a multipath analysis used in a deconvolution procedure applied to the coronary bed (Knopp et al., 1976). It is a convenient function for describing smooth unimodal density functions since it is rapidly computed using rational approximations to the integral of the Gaussian distribution and has parameters which are easily interpreted in terms of the central moments and therefore easy to use as shaping parameters (Bassingthwaighte, Ackerman, and Wood, 1966). The differential form of the density function *h(t)* is:

$$h(t) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{t-t_c}{\sigma}\right)^2} - \tau\frac{dh}{dt}$$

where σ and t_c are the standard deviation and centroid of a Gaussian function and τ is the time constant of a first order lag process. In the form of a convolution integral, the same function is:

$$h(t) = \int_0^t \frac{1}{\tau} \cdot e^{-\left(\frac{t-\lambda}{\tau}\right)} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{\lambda-t_c}{\sigma}\right)^2} d\lambda$$

where λ is a dummy variable of integration. The mathematical components, illustrated in the figure, should not be regarded as separate physical processes, but the overall function serves as a descriptor due to a combination of dispersive processes, the velocity profile, diffusion, and intravascular mixing.

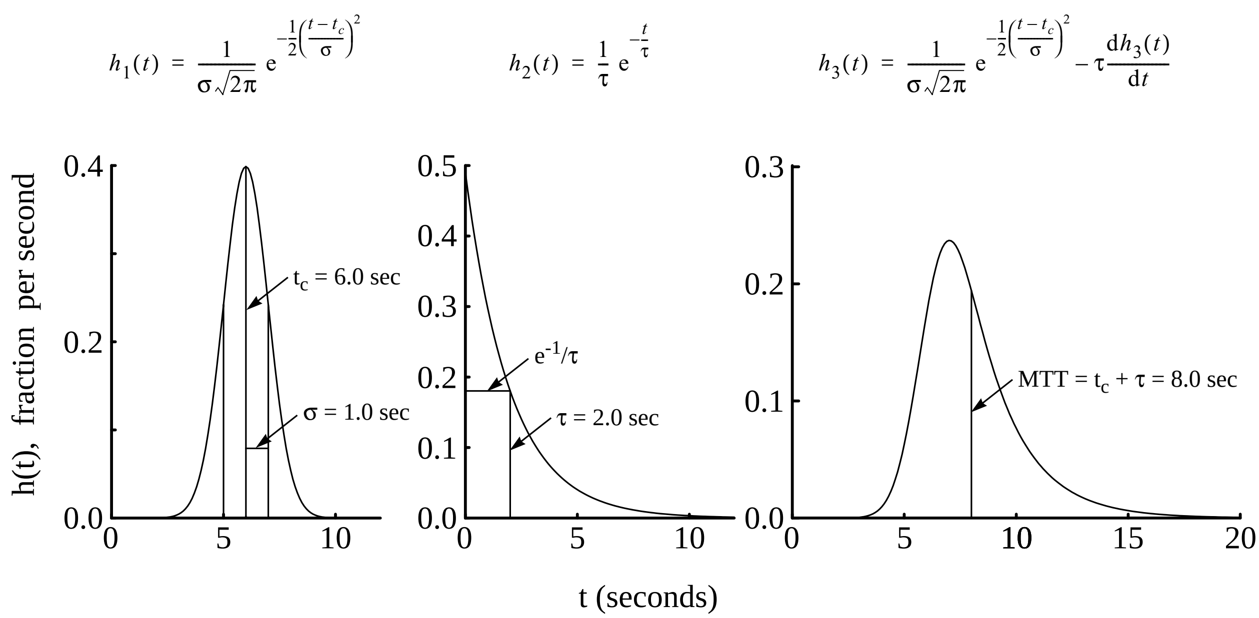

Figure 1: Diagram of the mathematical components of a lagged normal density curve (LAGNDC). *Left:* Gaussian density function with mean t_c and a standard deviation σ. *Middle:* Exponential density function with mean τ (and standard deviation τ). *Right:* The composite LAGNDC. (From Bassingthwaighte et al., 1966)

the moments of the density function are the sums of the moments of the individual components, apart from the zeroth moment, the area, which is unity. The mean, t, the variance π_2 (or the square of the standard deviation), and the third and fourth central moments about the mean, π_3 and π_4, are:

$$\bar{t} = t_c + \tau$$

$$\pi_2 = \sigma^2 + \tau^2 = SD^2$$

$$\pi_3 = 2\tau^2$$

$$\pi_4 = 3\sigma^2 + 6\sigma^2\tau^2 + 9\tau^4$$

These in turn provide the relative dispersion, RD, the standard deviation divided by the mean:

$$RD = \frac{\sqrt{\sigma^2 + \tau^2}}{t_c + \tau}$$

and the skewness, β_1, which is $\pi_3 / \pi_2^{3/2}$:

$$\beta_1 = \frac{2\tau^3}{(\sigma^2 + \tau^2)^{3/2}}$$

For symmetrical density functions $\beta_1 = 0$; positive β_1's indicate right skewing and negative β_1's left skewing. The maximum $\beta_1 = 2.0$, which puts all of the dispersion into the exponential component, so that $\sigma = 0$ and RD = τ / t.

The kurtosis, β_2, is π_4 / π_2^2:

$$\beta_2 = \frac{3\sigma^4 + 6\sigma^2\tau^2 + 9\tau^4}{(\sigma^2 + \tau^2)^2}$$

The kurtosis is a measure of “peakedness” of the distribution, being 3.0 for a Gaussian density function, greater for leptokurtic (high peaked) functions and less for platykurtic (flat-topped) functions. The kurtosis and higher moments contain no new information.

The calculation of the parameters from the moments is useful when one has only the moments of a transport function and wishes to define an appropriately shaped h(t). From the mean transit time, $\bar{t}$, the relative dispersion, RD, and the skewness, β_1, we obtain:

$$\tau = \mathrm{RD} \cdot \bar{t}(0.5\beta_1)^{1/3}$$

with the maximum $\tau = RD \cdot \bar{t}$ at $\beta_1 = 2.0$. For $\beta_1 < 2$, σ is calculable:

$$\sigma = [(\mathrm{RD} \cdot \bar{t})^2 - \tau^2]^{1/2}$$

and t_c is:

$$t_c = \bar{t} - \tau$$

The LAGNDC subroutine uses simpler models in three extreme cases: when $\tau < \sigma / 40$, a Gaussian distribution is returned; when $\sigma < \tau / 50$, an appropriately delayed exponential is returned; when the standard deviation is less than a quarter of a time step, $h(t) = \delta(t - \bar{t})$.

References

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