Introduction to mechanistic data-driven methods for engineering, mechanical science and mechanics of materials: difficulties in purely data-driven approaches for machine learning and some possible remedies

Prof. Wing Kam Liu, Walter P. Murphy Professor

Director of Global Center on Advanced Material Systems and Simulation (https://camsim.northwestern.edu/)

Northwestern University, w-liu@northwestern.edu

Students, Postdocs

Hengyang Li
Mahsa Tajdari
Satyajit Mojumder
Sourav Saha
Puikei Cheng
Orion L Kafka
Jiaying Gao
Cheng Yu
Dr. Zhengtao Gan

Collaborators

Greg Olson (NU, QuesTek)
NIST (Lyle Levine, Paul Witherell, Yan Lu)
CHiMaD (https://chimad.northwestern.edu/)
Jian Cao, Kori Ehmann
Greg Wagner
CT Wu, Zeliang Liu, and John Hallquist (LSTC)
Xianghui Xiao and Tao Sun (ANL)
John Sarwark (Feinberg NU, Lurie Hospital)

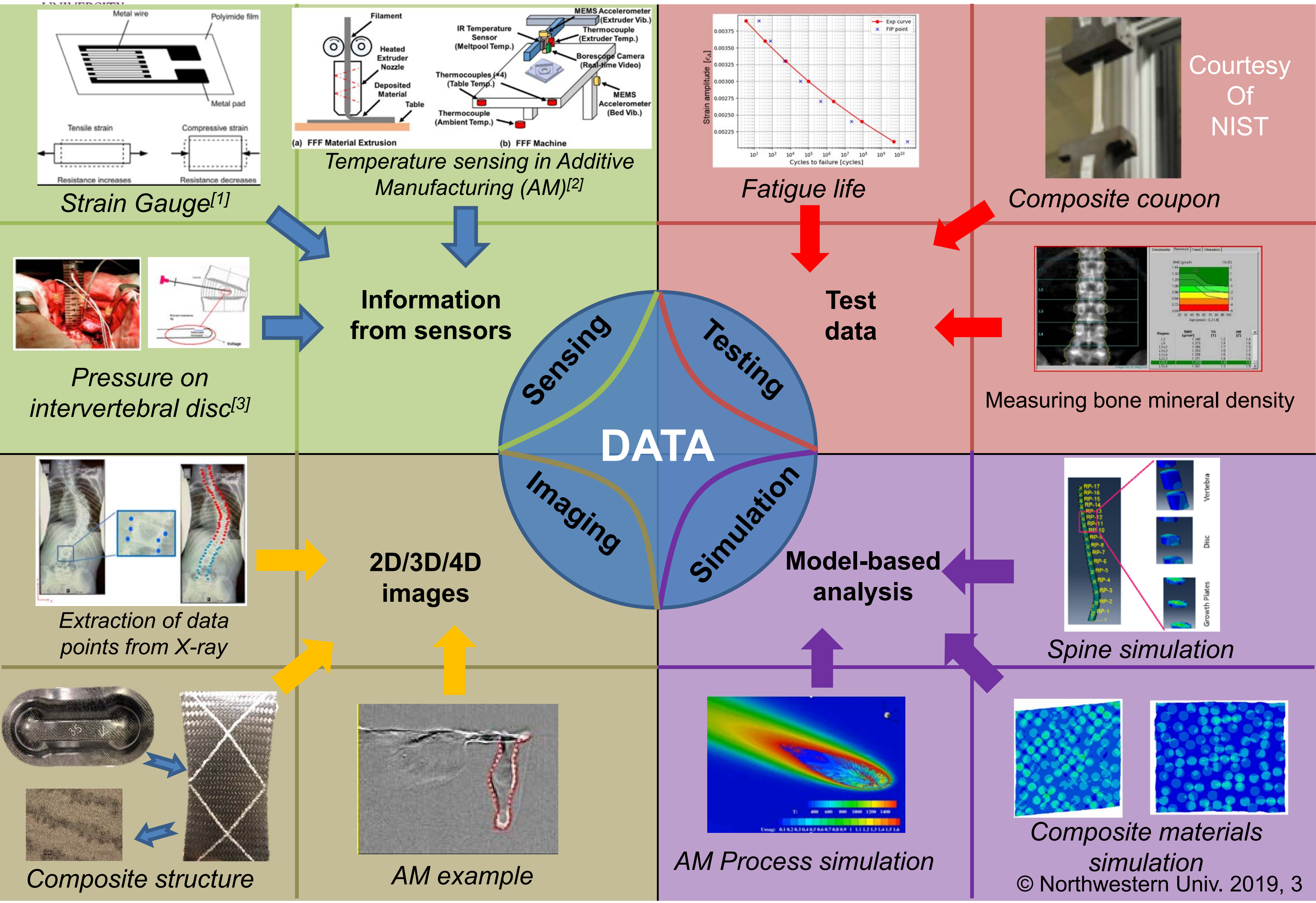
Outline

1. Motivation: sources of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

Outline

1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

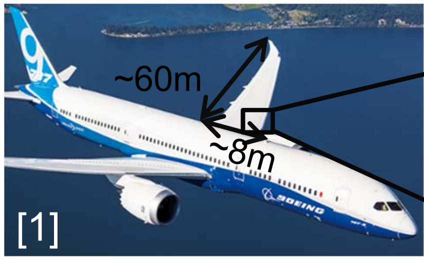
Multimodal data generation and collection



Data generation and collection in composite systems

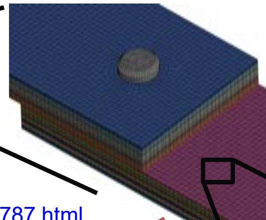
Data exist in multiple length scales for composite materials system

- The following sample cases are based on Unidirectional (UD) composite microstructure
- Local and average response of UD composite's Representative Volume Element (RVE) are of interest



[1] https://www.mhi.com/products/air/boeing_787.html

Single lap bolt joint



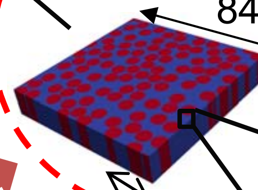
Health **monitoring** for structural performance

Woven composite

10.2 mm

UD composite

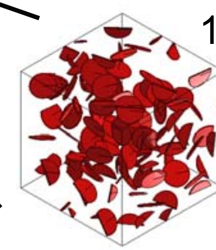
84 μm



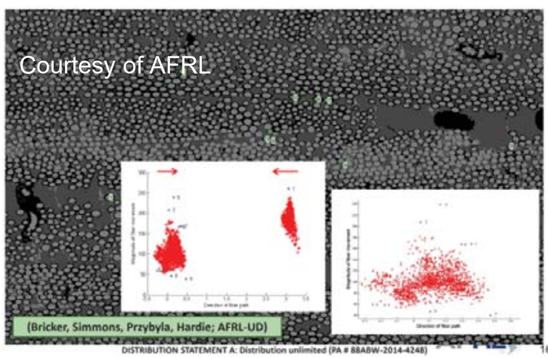
Fiber diameter: 7 μm

MoS_2 reinforced polymer matrix

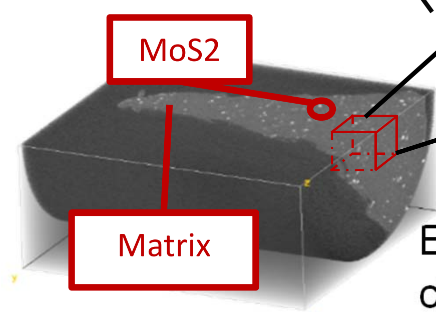
1 μm



Heterogeneous, non-stationary microstructure



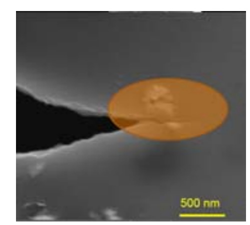
Anomaly detection for fiber orientation



Courtesy of AFRL

Experimental characterization and imaging

Interface fracture

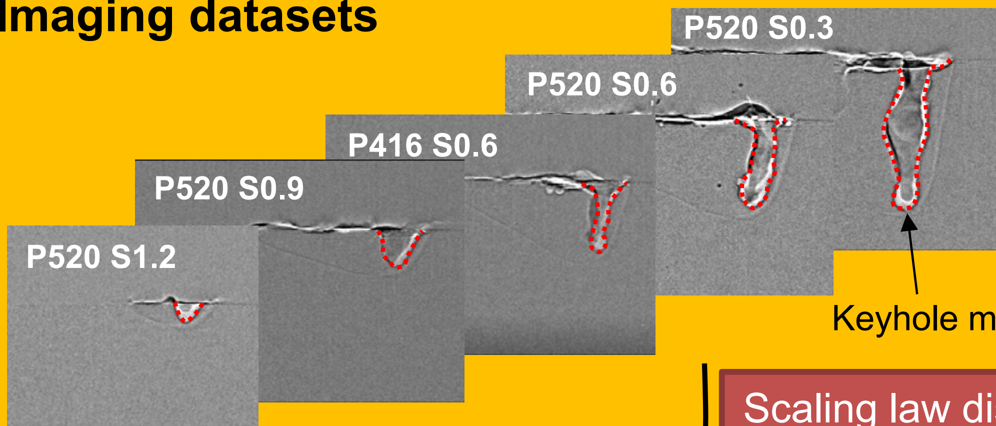


Courtesy of AFRL

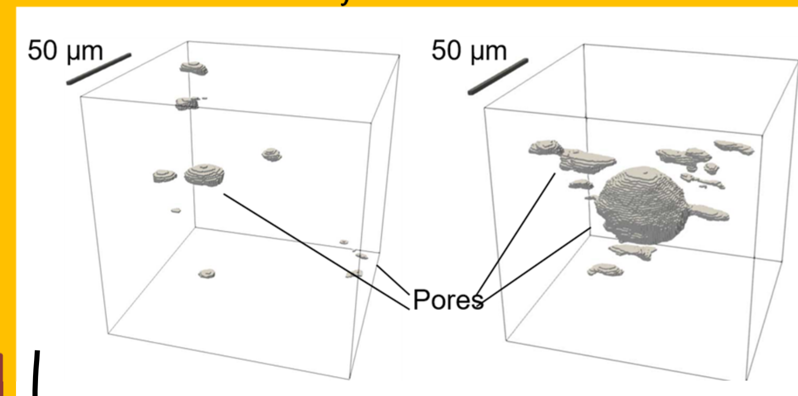
- Microstructure, material properties, structural performance.
- Information from four different scales are integrated to predict properties at part scale.

Approach: experiments and modeling motivated by data

Imaging datasets



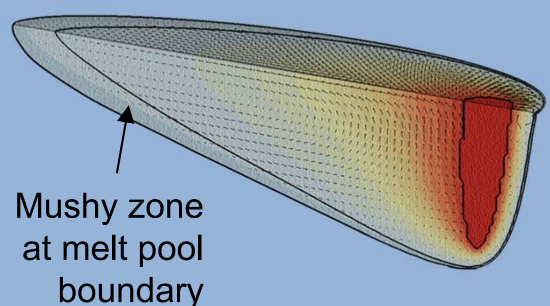
Porosity structure



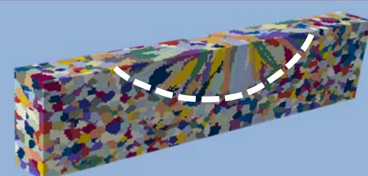
Scaling law discovery via machine learning

Image-based fatigue predictions

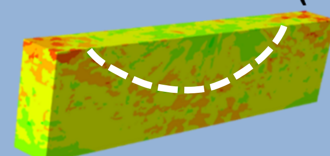
Data driven Process-structure-properties modeling



(Cannot be observed by in-situ experiments)

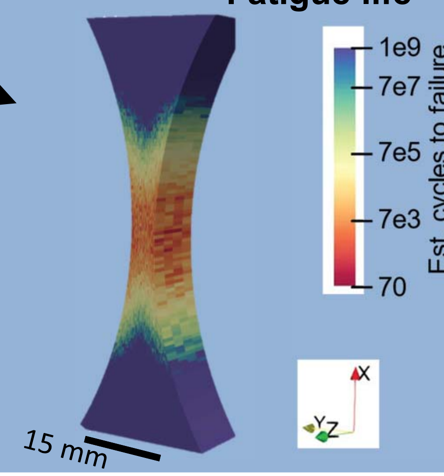


Cellular automaton (CA) modeling

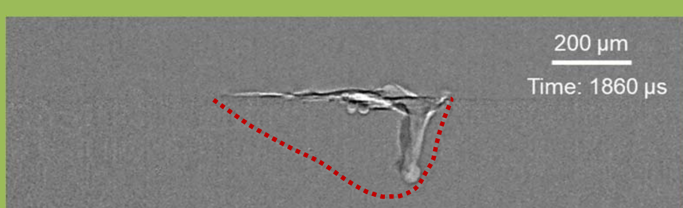


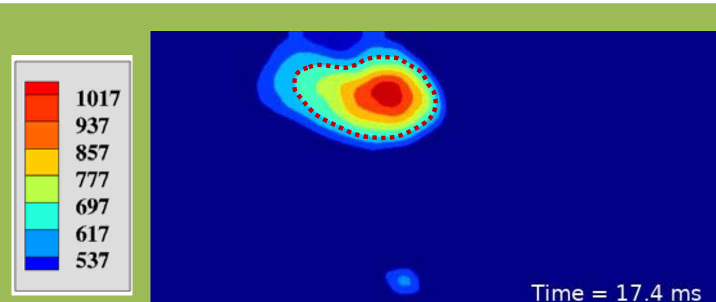
Grain-scale thermal-mechanical modeling

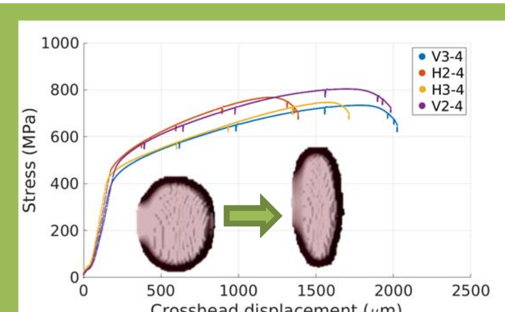
Fatigue life



Validation by experiments

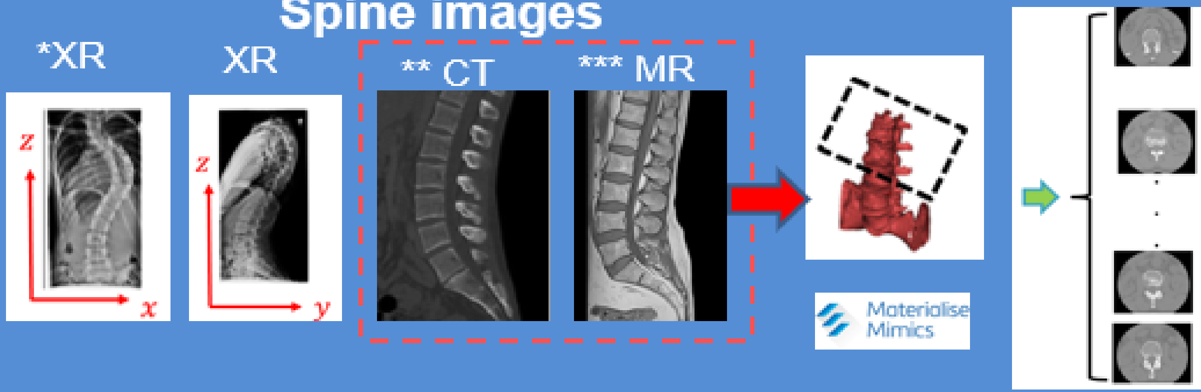




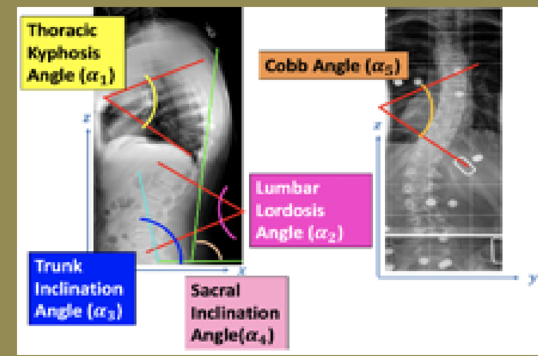


Data generation and collection in Adolescent Idiopathic Scoliosis (AIS)

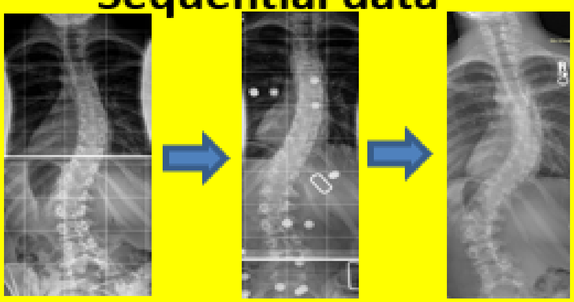
Spine images



Spinal angles (3D)

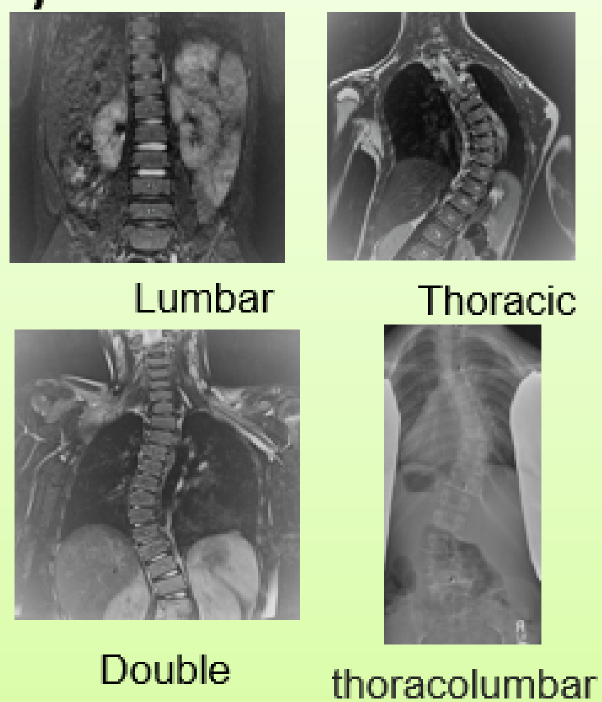


Sequential data



Age: t_1 months t_2 months t_3 months

All curvature types proposed by Scoliosis Research Society



Bone mineral density

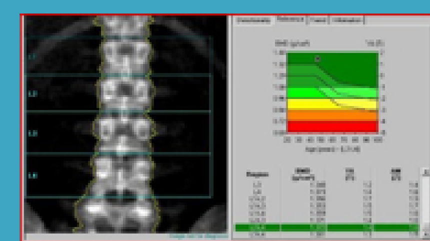
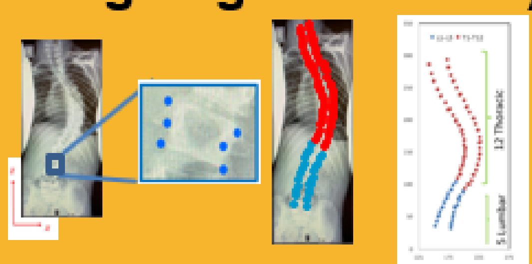
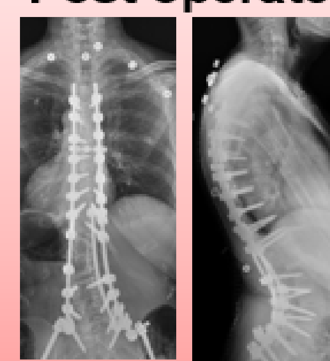


Image segmentation (Assigning landmarks)



Post operated data



Spinal fusion

*XR: Xray ** CT: Computerized Tomography, ***MR: Magnetic Resonance, courtesy: Lurie Children's Hospital of Chicago

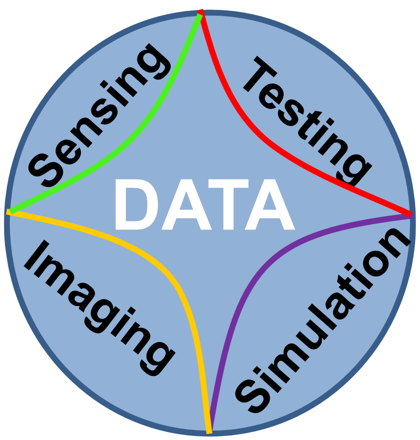
Outline

1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

Interpretation of data in mechanical science and engineering

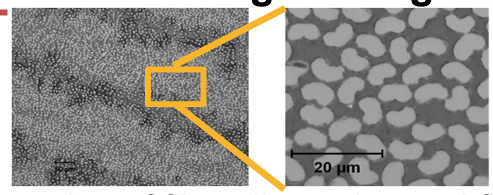
NORTHWESTERN UNIVERSITY

Data analysis using Machine Learning



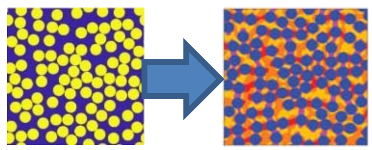
Means

Feature Engineering



Extraction of fiber dimension and fiber distribution from UD cross section

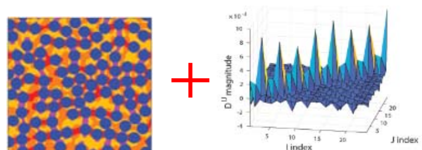
Dimension Reduction



600 x 600 voxels

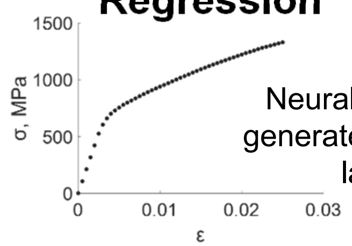
4 clusters

Reduced Order Models



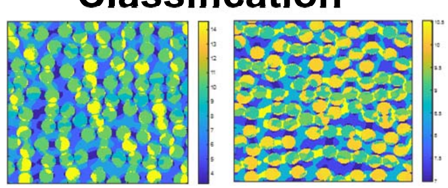
Interaction tensor

Regression



Neural Network generated material laws

Classification



Debonding

No debonding

Goals

To extract meaningful data

To reduce degrees of freedom

To speed up the computation

To discover hidden relationships

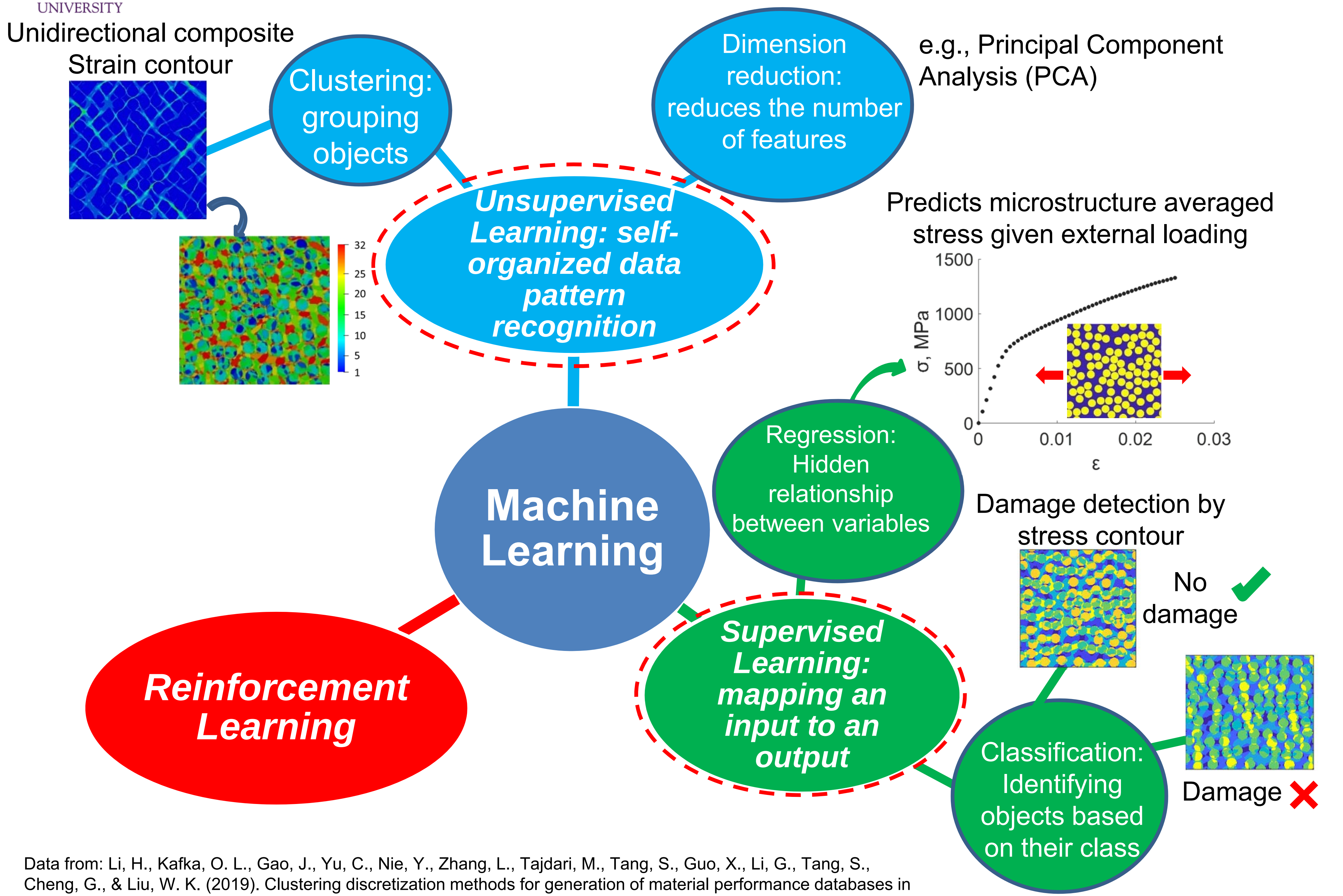
To categorize data

Knowledge for applications e.g. material systems design and discovery

Outline

1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

Three types of machine learning in mechanical science and engineering



Data from: Li, H., Kafka, O. L., Gao, J., Yu, C., Nie, Y., Zhang, L., Tajdari, M., Tang, S., Guo, X., Li, G., Tang, S., Cheng, G., & Liu, W. K. (2019). Clustering discretization methods for generation of material performance databases in machine learning and design optimization. Computational Mechanics, 1-25.

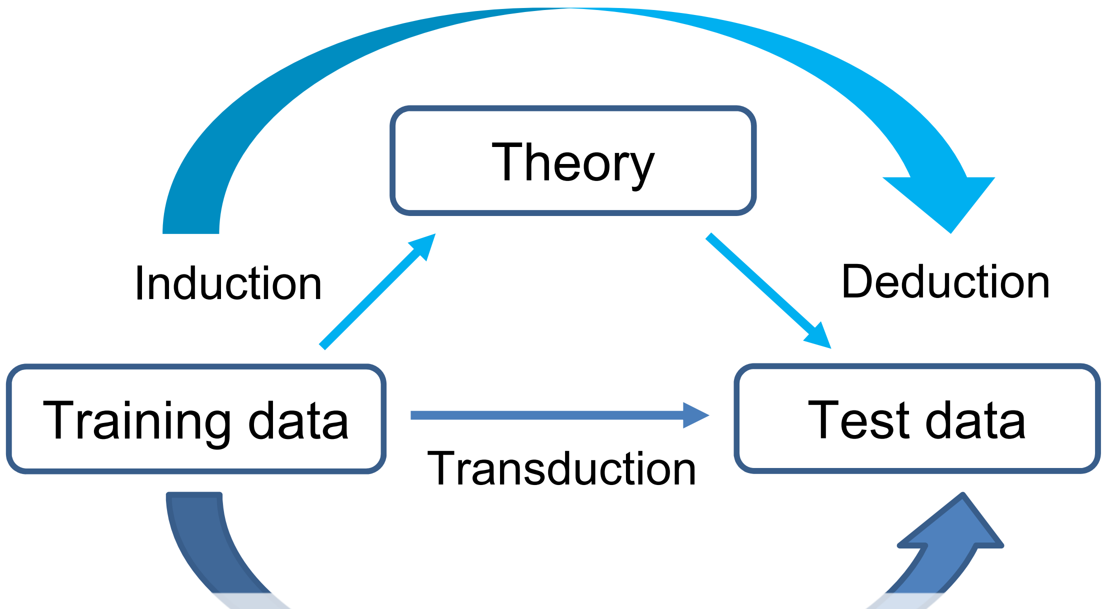
Data-Science: Transduction

Data-science is the “fourth paradigm” of science (empirical, theoretical, computational, data-driven) [21]

Conventional methods

Induction: specific observations to general theory (bottom-up)

Deduction: general theory to testable observations (top-down)



Machine/Deep learning:

Transduction: learn from given data to apply to new data [22]

Relevant concepts in data science

Machine Learning

❑ A program or system that builds (trains) a predictive model from input data

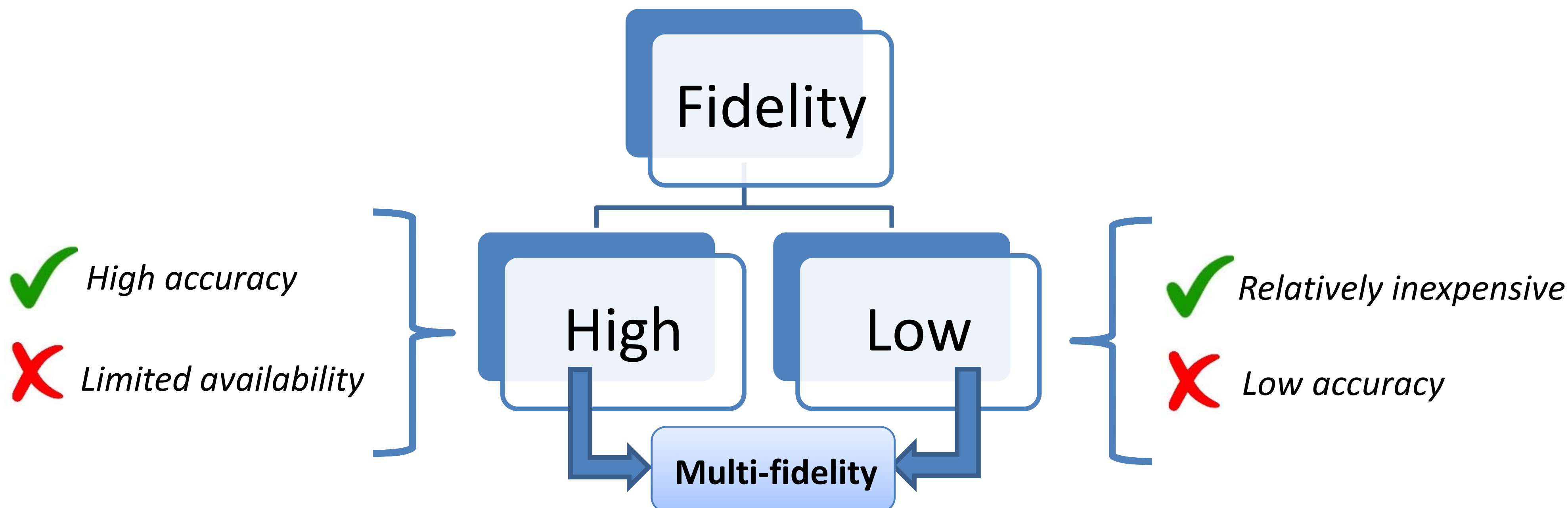
Dimensionality

❑ Feature dimensionality: The number of features for each data point

❑ Input dimensionality: The total number of data points

Fidelity

❑ Quality of faithfulness of data



Relevant concepts in data science (cont)

Database

- A collection of rows or dataset with one or more features.

Features

- Individual independent variables defining characteristics of a data set.
- Informative and non-redundant data.

Feature engineering

- Process of determining which **features** might be useful and converting raw data into said features.

Dimension reduction

- Process of decreasing the number of dimensions representing a feature.

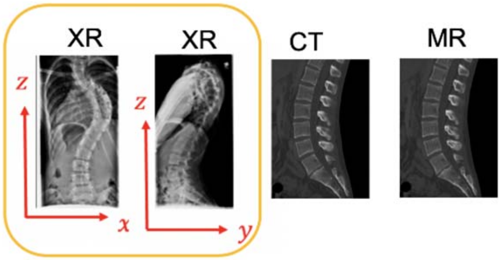
Objective function

- The mathematical formula or metric that a model aims to optimize.

Courtesy: Google developers https://developers.google.com/machine-learning/glossary/#m

Illustration of relevant concepts for AIS*

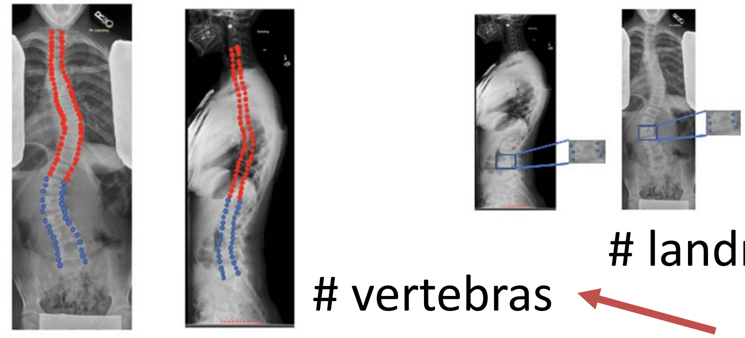
NORTHWESTERN UNIVERSITY



Data gathered using 2D XR images

Source

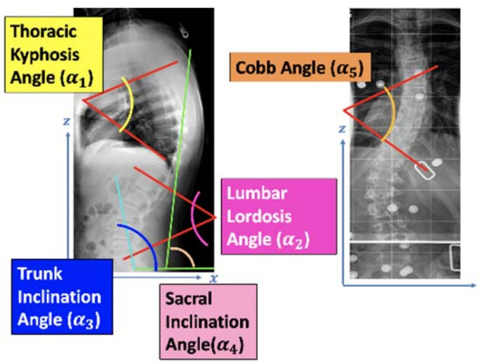
Database Generation



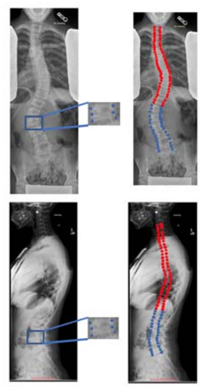
vertebras

landmarks

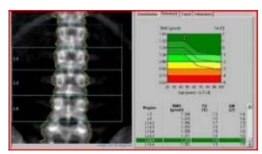
Dimensionality of data(Ns): 17x2x6



Spinal Angles



Co-ordinate



BMD**

Features

- Age
- Co-ordinates
- Spinal Angle
- Stress
- BMD

Dimensionality of features

Features	Data points					
	1	2	3	.	.	Ns
X						
σ	.	.	.	.	.	.
α	.	.	.	.	.	.
t	.	.	.	.	.	.
Δt						
BMD						

Fidelity

High
- Co-ordinates from x-ray
- Age, test frequency

Low
- Spinal angles
- Stress

X = Vector of input coordinates of a landmark $[X_1 \quad X_2 \quad X_3\ ]$

σ = Stress vector $[\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{23}\ \sigma_{31}]$

α = Global angel (Spinal Angles) vector $[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ ]$

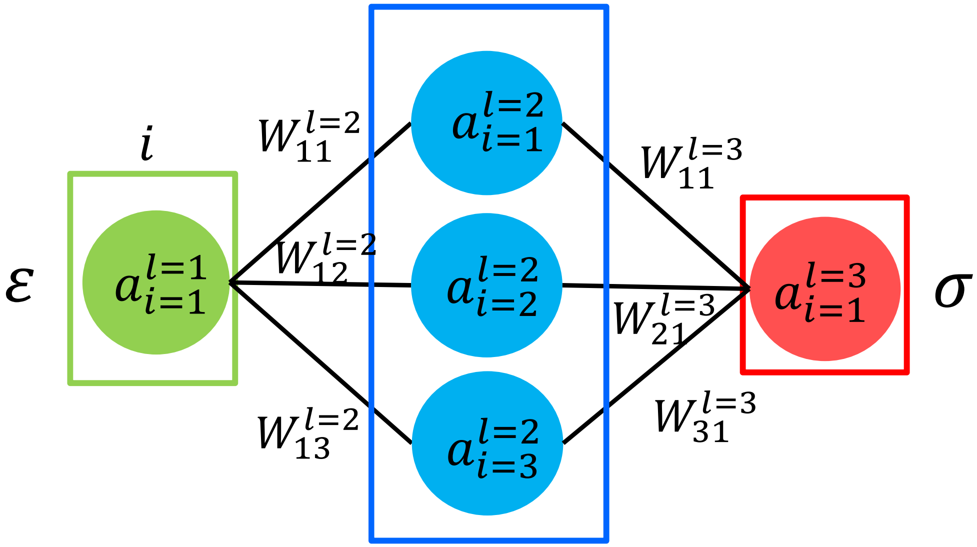
t = Age of the patient

Δt = age variance between target age and current age (month)

*Adolescent Idiopathic Scoliosis **Bone Mineral Density

Basic concepts of artificial neural network (ANN)

Objective: To learn hidden relationship between input and output



Layer $l = 1$ **Layer** $l = 2$ **Layer** $l = 3$

Input layer **Hidden layer** **Output layer**

Sample input data

	$\boldsymbol{\varepsilon}$	$\boldsymbol{\sigma}$ **(MPa)**
Data point 1	0.1	20
Data point 2	0.2	38.6
...		

Assume $\sigma^* = 20$

a_i^l: value of each neuron

W_{ij}^l: weights connecting neurons

b_i^l: bias on each neuron

Optimization problem:

$$minimize\ Error: E = \frac{1}{2}(\sigma^* - a_{i=1}^{l=3})^2$$

Gradient descent[1] **:**

$$\Delta W_{11}^{l=3} = \alpha \delta a_{i=1}^{l=2} \qquad \Delta W_{11}^{l=2} = \alpha \delta W_{11}^{l=3}\ a_{i=1}^{l=1}$$

$$\Delta W_{21}^{l=3} = \alpha \delta a_{i=2}^{l=2} \qquad \Delta W_{12}^{l=2} = \alpha \delta W_{12}^{l=3}\ a_{i=1}^{l=1}$$

$$\Delta W_{31}^{l=3} = \alpha \delta a_{i=3}^{l=2} \qquad \Delta W_{13}^{l=2} = \alpha \delta W_{13}^{l=3}\ a_{i=1}^{l=1}$$

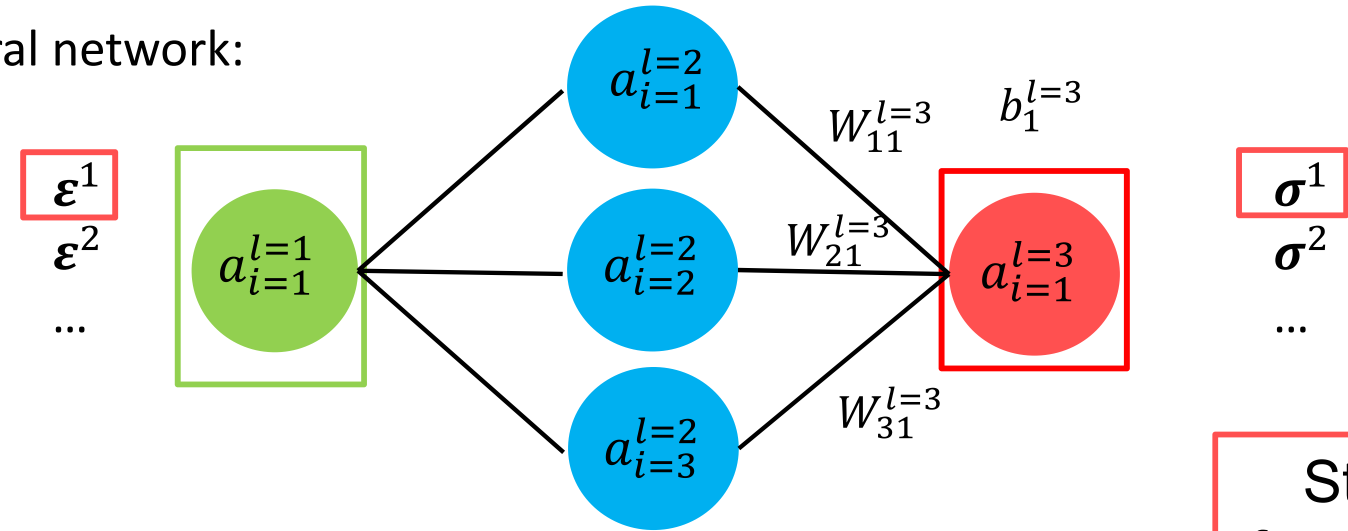
$\sigma^*: target\ value$ $\quad \alpha: learning\ rate$

$\delta = (\sigma^* - a_{i=1}^{l=3})$

[1] Boyd, S., & Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press.

Example of training Neural Network (NN): learning back-propagation

Neural network:



Start training from data point 1

Layer $l = 1$ **Layer $l = 2$** **Layer $l = 3$**

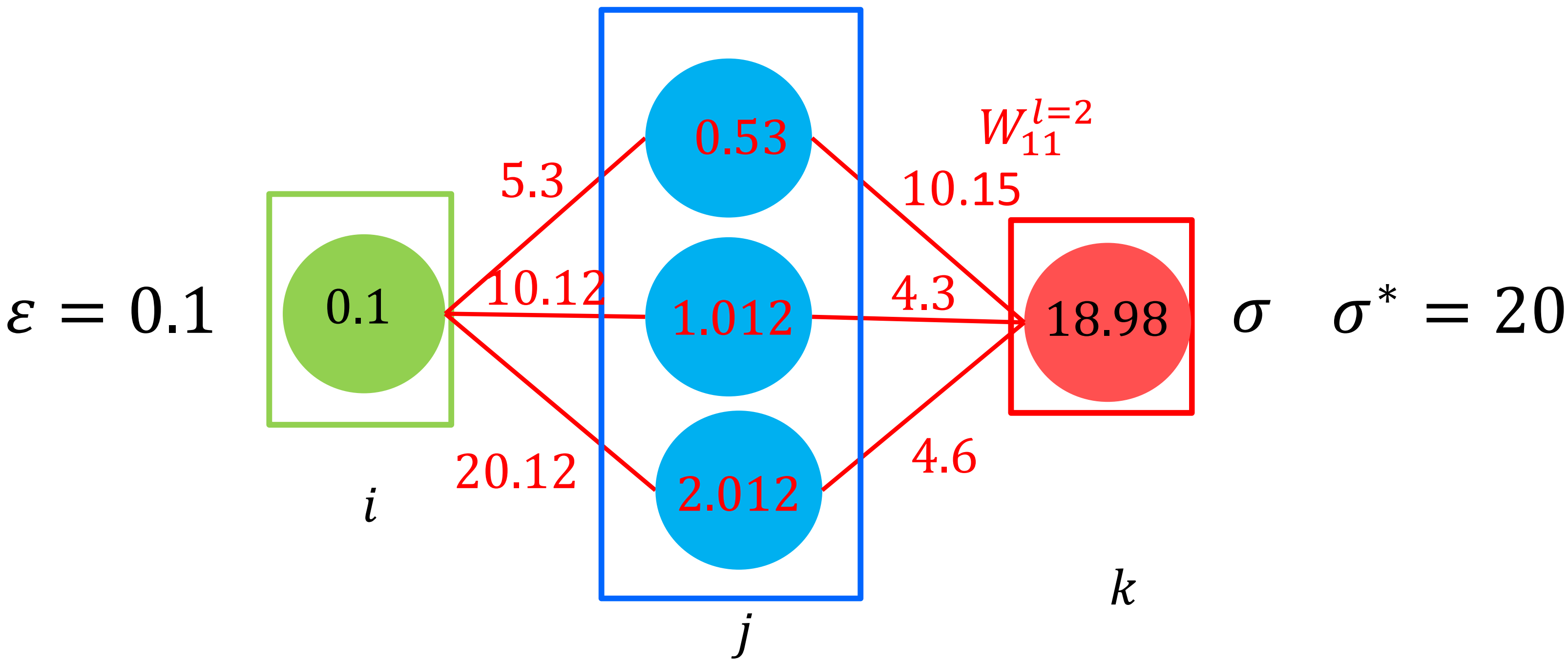
Input layer **Hidden layer** **Output layer**

$$a_1^{l=3} = f\left(\sum_{i=1}^{3} W_{i,1}^{l=3} a_i^{l=2} + b_1^{l=3}\right)$$

$f(X)$: $activation\ function,$

e.g. ReLU function, $f(X) = \max(0, X)$

Check error, and iterate for convergence



$$Error: E = \frac{1}{2}(\sigma^* - a_{k=1}^{l=3})^2$$

$$E = 4.5 \rightarrow 0.5202$$

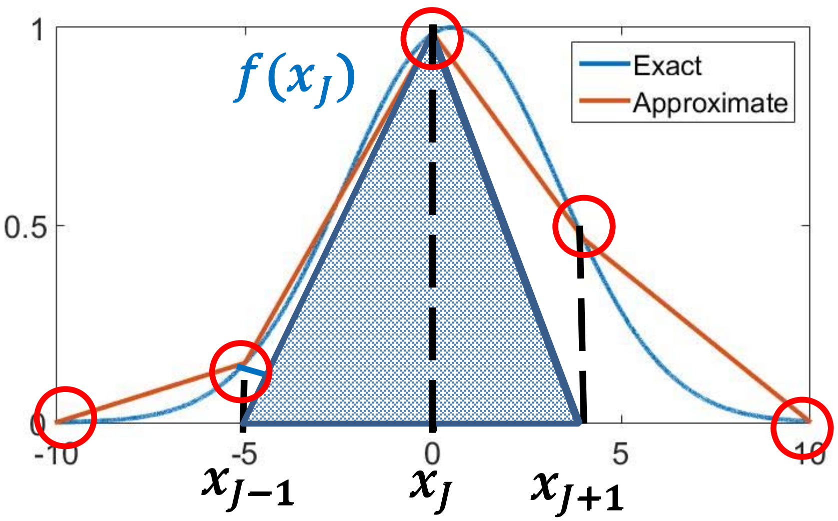
The error will reduce by iteration, finally

$$E \leq E^*, convergence$$

Repeat for all data points until error is minimized

Neural Network as interpolation function[1]

We often use a linear interpolation function $f^h(x)$ to approximate any continuous function $f(x)$.



- Goal function $f(x)$:
  e.g. $f(x) = \exp(-(x-0.5)^2/16)$ (blue line)
- Approximate function: N_f Nodes
  e.g. $\sum_{J=1}^{N_f=5} f(x_J)N(x;x_J)$,
  $\{x_1, x_2, \dots, x_5\} = \{-10, -5, 0, 4, 10\}$
  (Orange line)
- Assume linear shape function

$ReLU$: Rectified linear unit, usually in form: $\max(0, X)$

$$\sum_{J=1}^{N_f=5} f(x_J)N(x;x_J) = \sum_{J=1}^{N_f=5} \underbrace{T_J[x; ReLU, x_J, x_{J-1}, x_{J+1}]}_{\text{Shape function approximated by NN[2, 3]}} \boldsymbol{f(x_J)}$$

Shape function approximated by NN[2, 3] $\Leftrightarrow G^N(x; \{\boldsymbol{w}\}_J, \{\boldsymbol{b}\}_J)$

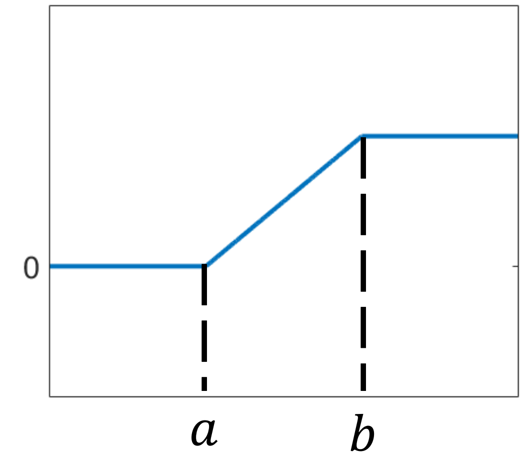
[1] Zhang, L., Yang, Y., Li H., Gao J., Reno D., Tang S., Liu W.K. Neural network finite element method, in preparation
[2] Approximation by superpositions of a sigmoidal function, by George Cybenko (1989).
[3] Multilayer feedforward networks are universal approximators, by Kurt Hornik, Maxwell Stinchcombe, and Halbert White (1989).

Unpublished results

Proof: NN for 1D shape function approximation

Lemma 1 The continuous piece-wise linear function

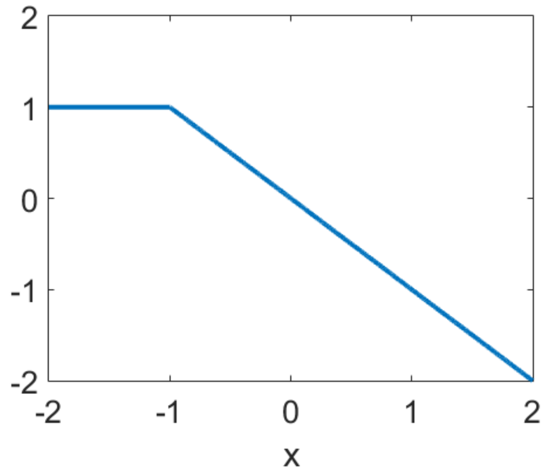
$$B(x;a,b)=\begin{cases} 0 & x<a \\ x-a & a\le x\le b \\ b-a & x>b \end{cases},$$



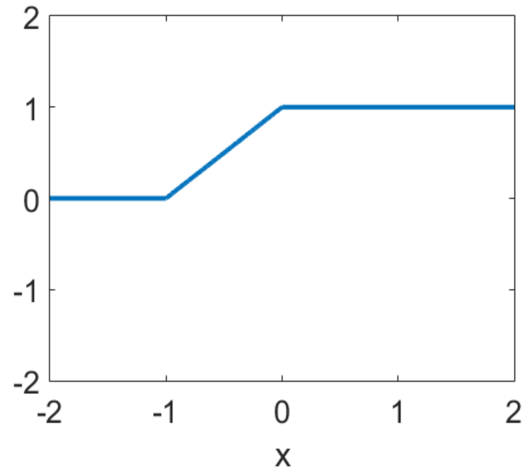
can be represented by neural network as

$$-ReLU(-ReLU(x-a)+b-a)+b-a$$

e.g. a=-1, b=0



Step 1: $-ReLU(x+1)+1$



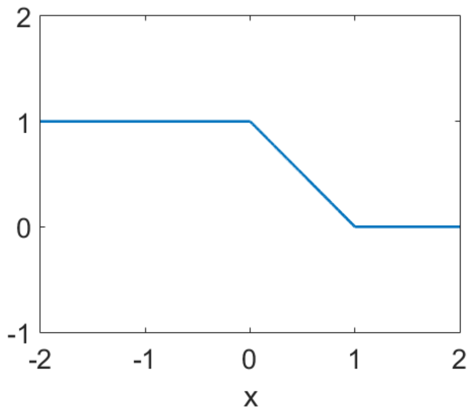
Step 2: $-ReLU(-ReLU(x+1)+1)+1$

Zhang, L., Yang, Y., Li H., Gao J., Reno D., Tang S., Liu W.K. Neural network finite element method, in preparation

Unpublished results

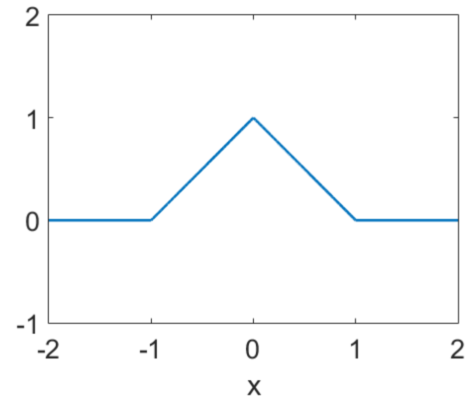
Proof: NN for 1D shape function approximation

For 1D linear basis function, take the reflection to construct the right part and then combine these two parts.



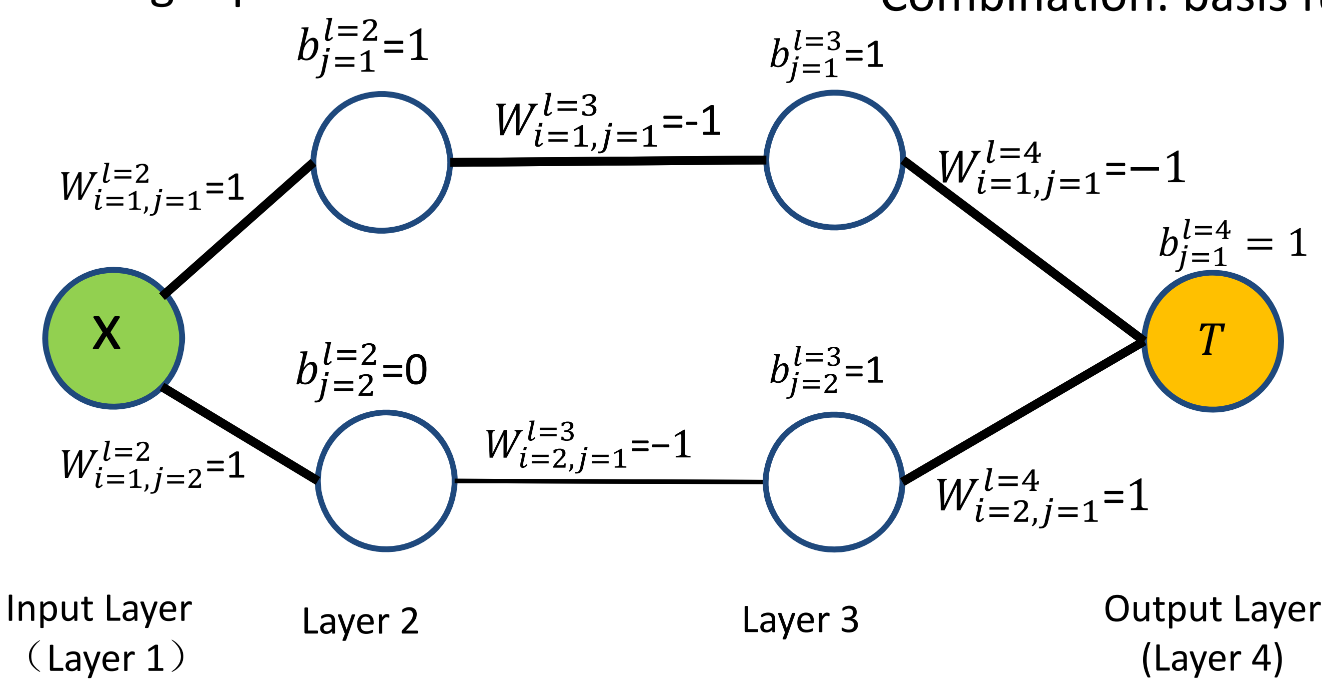
Step 3: $ReLU(-ReLU(x)+1)$

Reflection: right part



Step 4: $-ReLU(-ReLU(x+1)+1)+1$
$+ReLU(-ReLU(x)+1)$

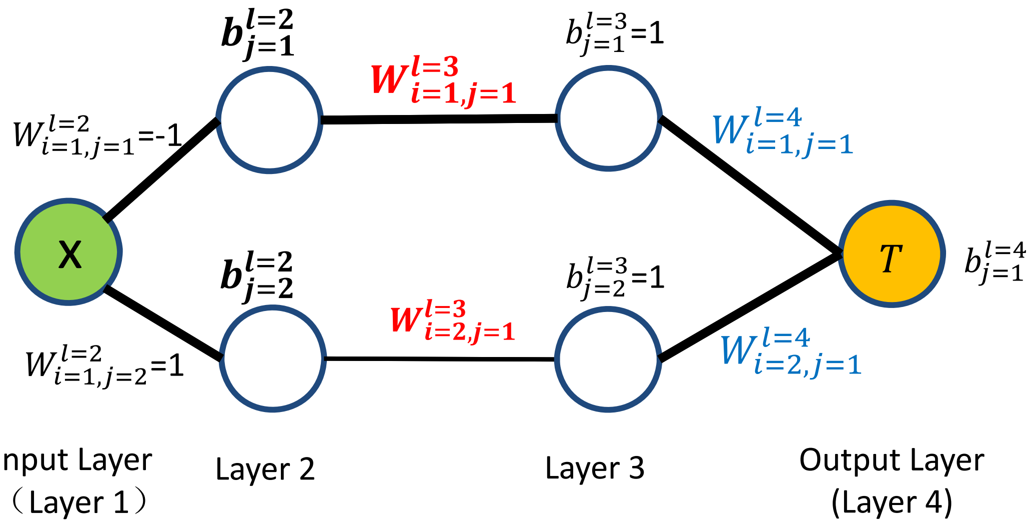
Combination: basis function



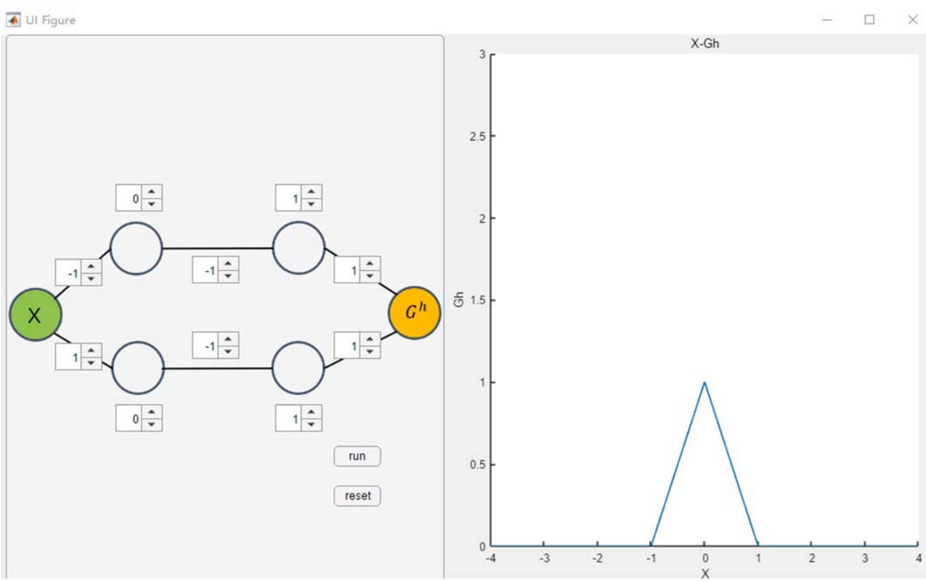
Unpublished results

NORTHWESTERN UNIVERSITY

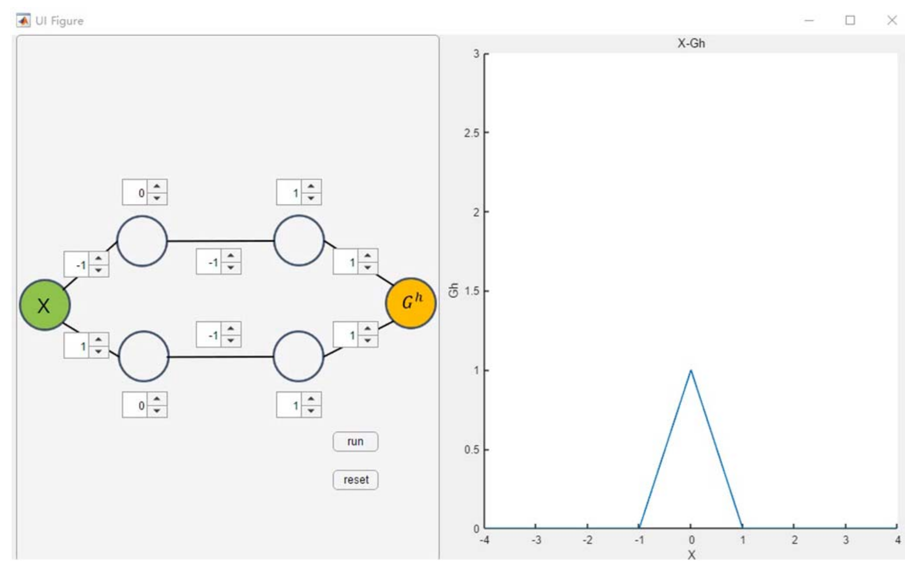
Effect of weights and biases on the output



	X	T
Data point 1	1	0
Data point 2	0.5	0.3
…		



Change in bias $b^{l=2}_{j}$, changes **the location**



Change in weights $W^{l=3}_{i=1,j}$, changes **the slope**

Zhang, L., Yang, Y., Li H., Gao J., Reno D., Tang S., Liu W.K. Neural network finite element method , in preparation

Unpublished results

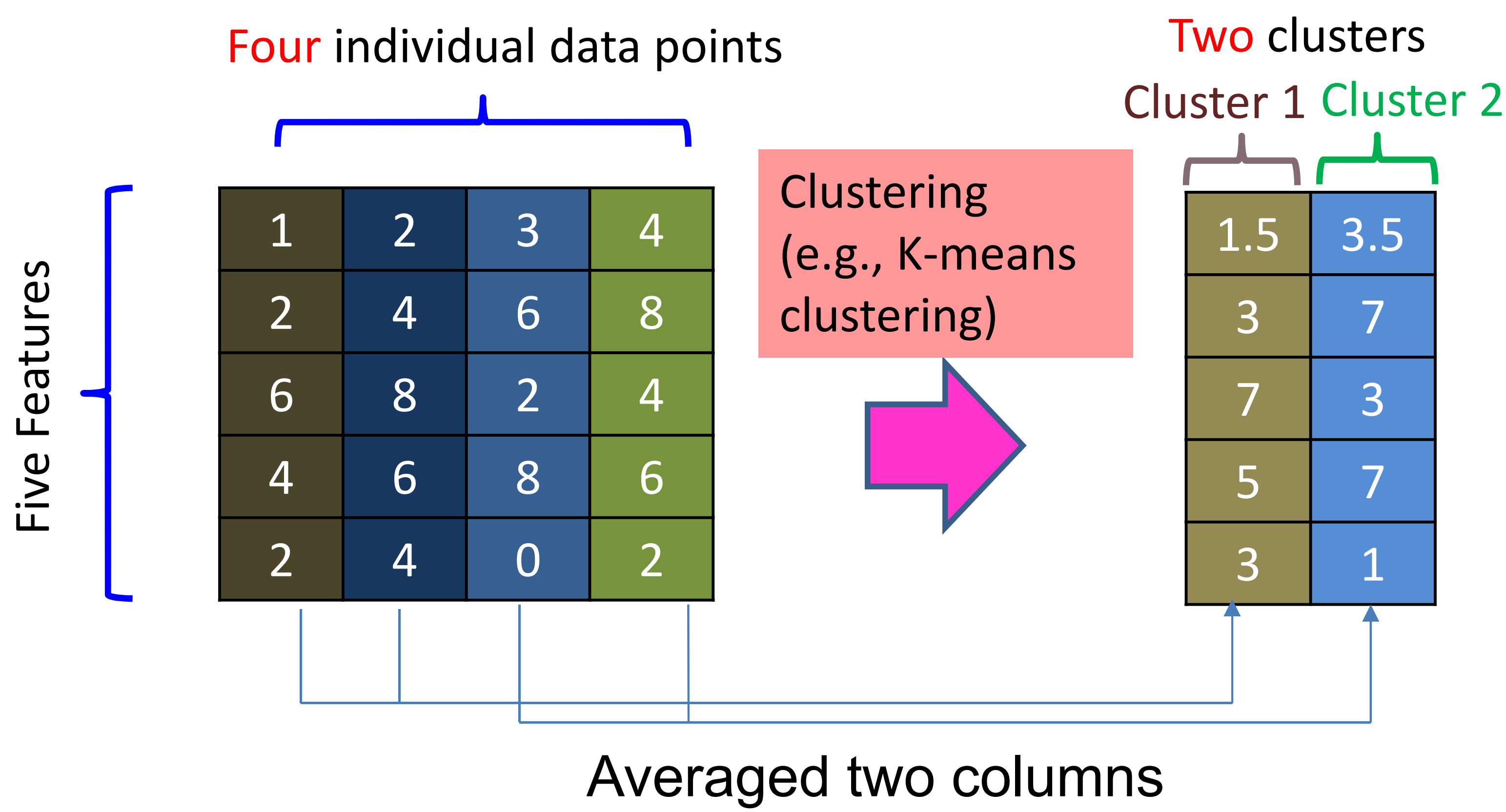
NORTHWESTERN UNIVERSITY

Outline

1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

A simple illustration on unsupervised learning for clustering

Objective: Group 4 data points (each having five features) into 2 clusters



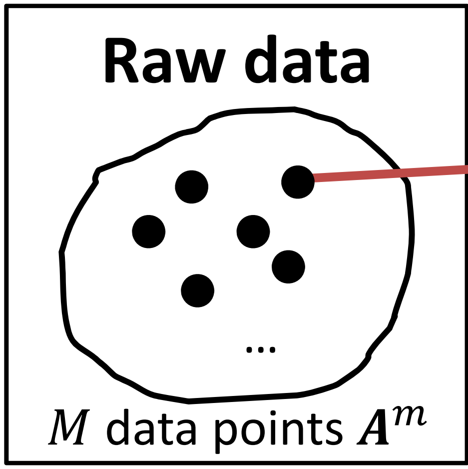
1	2	3	4
2	4	6	8
6	8	2	4
4	6	8	6
2	4	0	2

1.5	3.5
3	7
7	3
5	7
3	1

Clustering: Reduces the data dimensionality

Concept of K-means Clustering

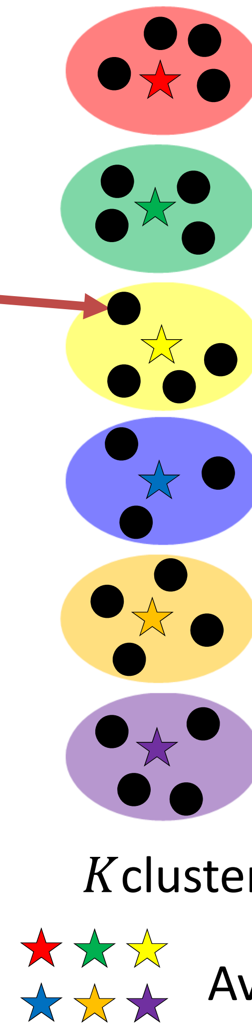
How does K-means clustering work ?



N dimensional

$\boldsymbol{A}^m = [A_1^m, A_2^m \dots A_N^m]$

Data point $\boldsymbol{A}^m$



$\boldsymbol{S}^1$, k=1

$\boldsymbol{S}^2$, k=2

$\boldsymbol{S}^3$, k=3

$\boldsymbol{S}^4$, k=4

$\boldsymbol{S}^5$, k=5

$\boldsymbol{S}^6$, k=6

K clusters

Average points $\boldsymbol{W}^k$

k: index of clusters

m: index of data points

- Cluster:
  - Points with most similar values
  - Has one average point: mean average of nearby data points
- Objective:

Minimize total distance between each average point and the data points within its cluster.

Mathematically: $$minimize: \sum_{k=1}^{K} \sum_{\boldsymbol{A}^m \in \boldsymbol{S}^k} \left\| \boldsymbol{A}^m - \boldsymbol{W}^k \right\|^2$$

$\boldsymbol{A}^m$: Data point in cluster $\boldsymbol{S}^k$

$\boldsymbol{W}^k$: Average point in cluster $\boldsymbol{S}^k$

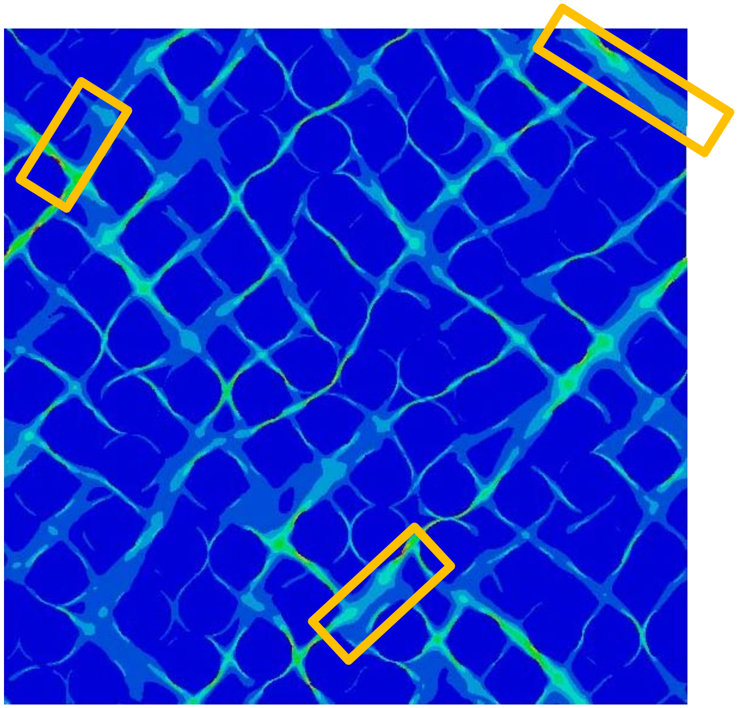
$\| \quad \|$: Euclidean distance

Watt, J., Borhani, R., & Katsaggelos, A. K. (2016).Machine learning refined: foundations, algorithms, and applications. Cambridge University Press.

K-means clustering for Unidirectional (UD) composite

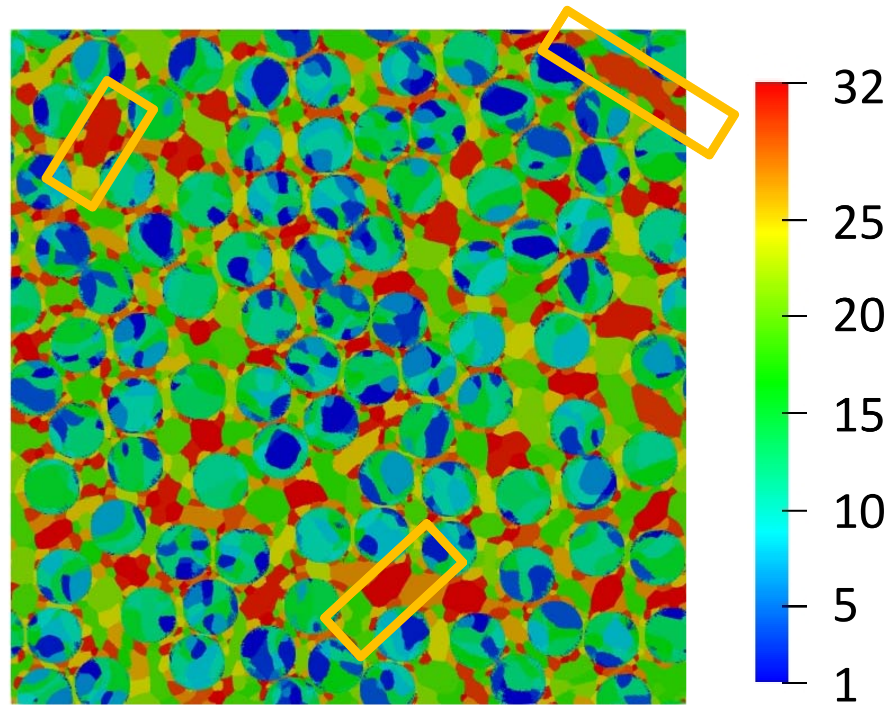
Grouping local material points in the microstructure based on strain responses (or other quantities, such as effective plastic strain)

Strain distribution



2D microstructure with 600 by 600 voxels

Cluster distribution based on strain intensity

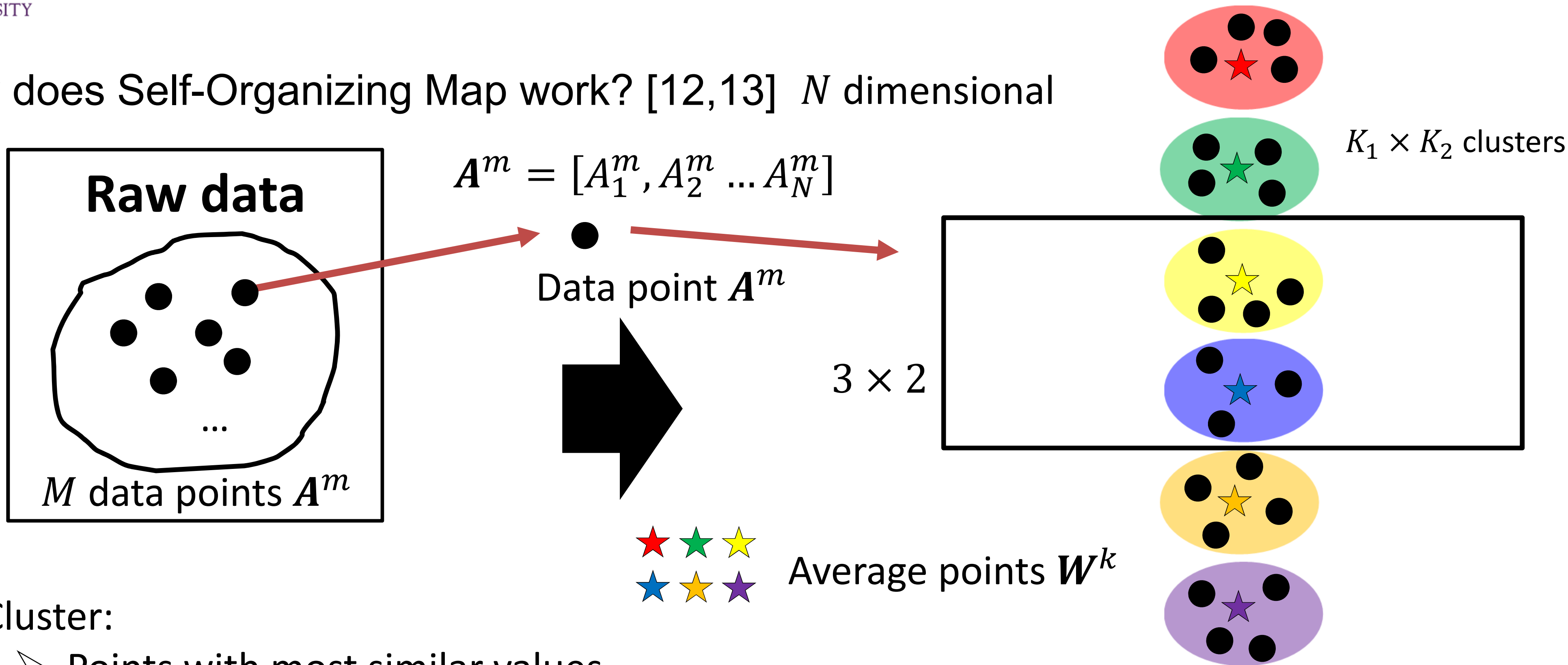


2D microstructure with 32 clusters

- The strain field, originally represented by 360,000 voxels, is now represented by 32 clusters
- The strain patterns are adequately captured by the clusters

Self-Organizing Map (SOM) - Concept

How does Self-Organizing Map work? [12,13]



- Cluster:
  - Points with most similar values
  - Has one average point: weighted average of nearby data points
- Objective:

Distribute all data points into **a map of $K_1 \times K_2$** clusters so that the dissimilarity within a cluster is minimized**, and the dissimilarity between clusters with nearby indexes is minimized**

$$minimize: \sum_{k=[1,1]}^{[K_1,K_2]} \sum_{\boldsymbol{A}^m \in \boldsymbol{S}^k} \sum_{k'=[1,1]}^{[K_1,K_2]} h(||k-k'||) \left\|\boldsymbol{A}^m - \boldsymbol{W}^k\right\|^2$$

$||k-k'||$: Euclidean distance between clusters' indexes

$h(||k-k'||)$: Gaussian kernel function

Sample clustering results from K-means and SOM

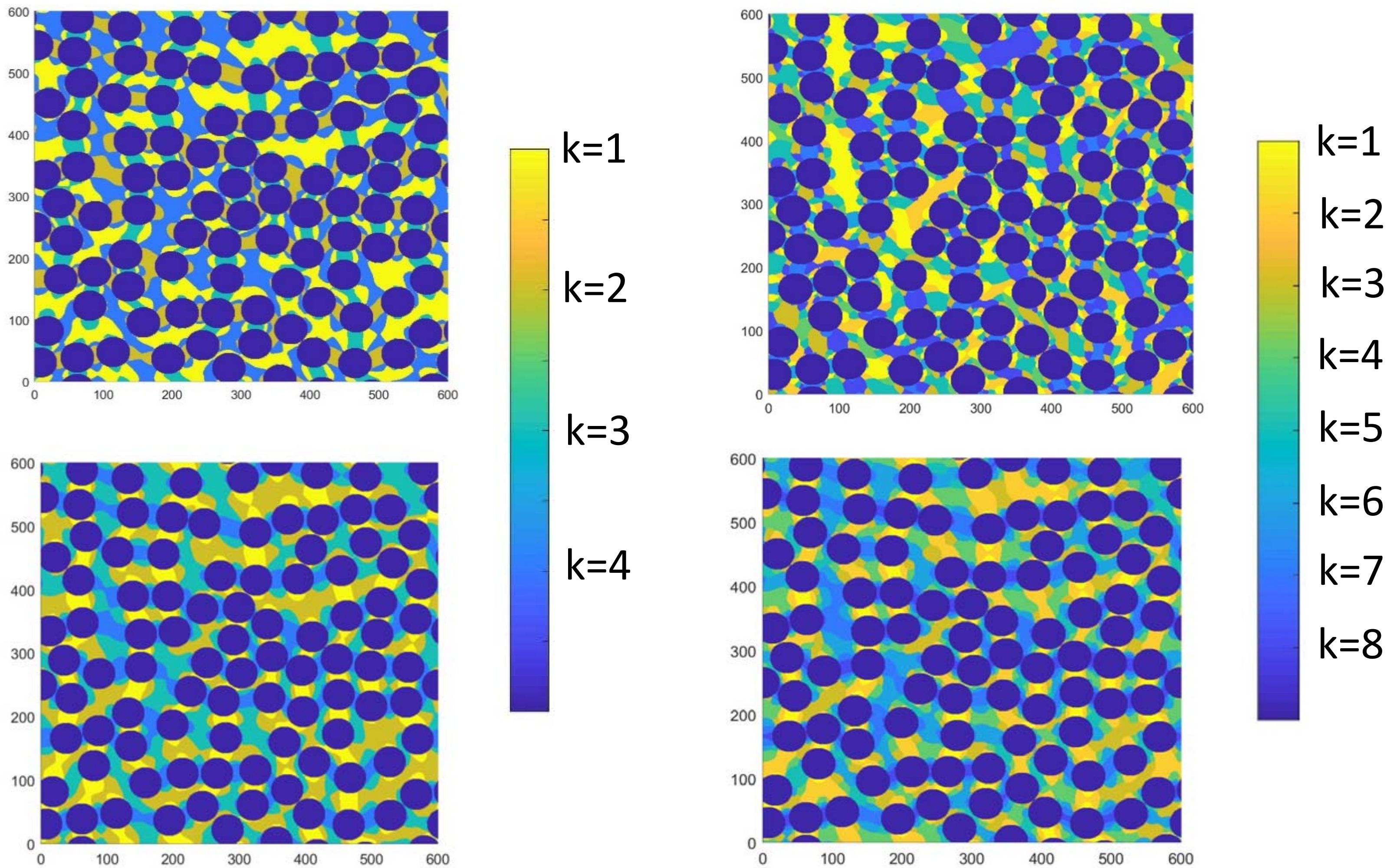
NORTHWESTERN UNIVERSITY

Unidirectional UD fiber composite (plane strain condition)

K = 4 clusters (matrix) | K = 8 clusters (matrix)

K-means (Matrix phase)

Self-organizing map (Matrix phase)

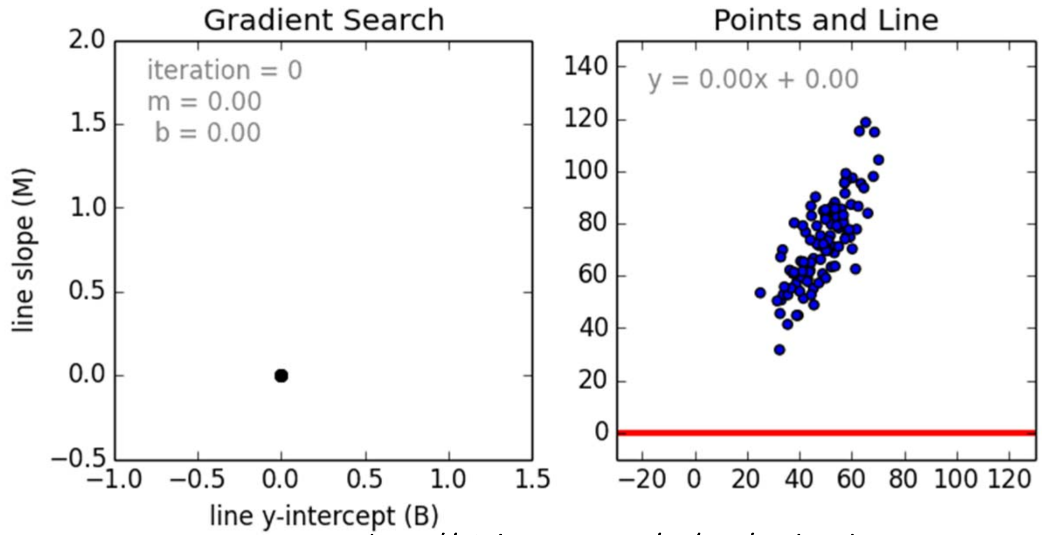


SOM provides orderly ranking of clusters, feature indicators, and physical insights, e.g. strain distribution, damage

Supervised learning

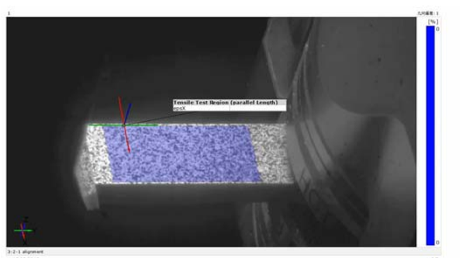
- Supervised learning establishes the hidden relationship between the input and output data.

Quick overview of Supervised Learning



Courtesy: http://aiobserve.com/AI/ML/31.html

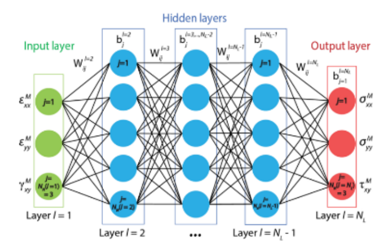
- Can predict the material law from input and output strain-stress data.



Sample [1]

	ε	σ (Mpa)
Data point 1	0.1	20
Data point 2	0.2	38.6
…		

experiment



Training

σ = NN(ε)

Prediction

- Used for – Regression and Classification

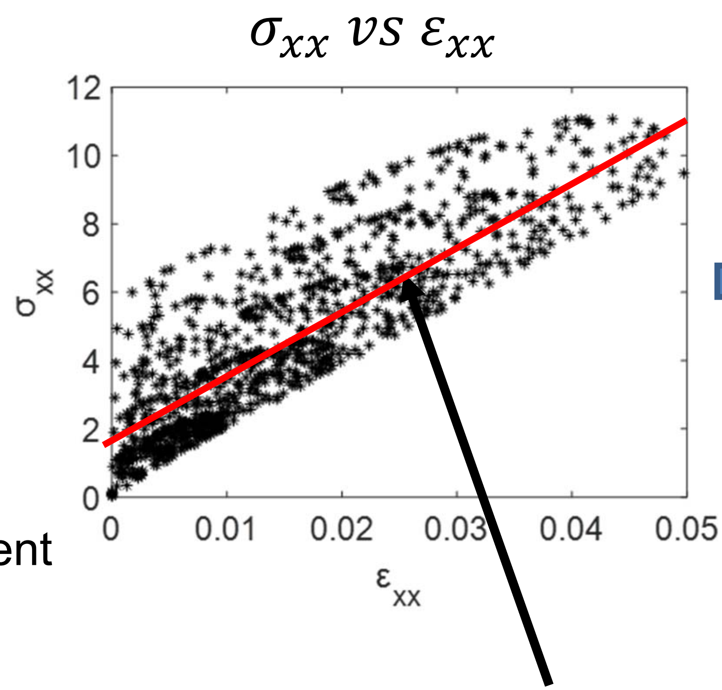
[1] Tang, S., Zhang, G., Yang, H., Guo, X., Li, Y., & Liu, W. K. (2019). MAP123: A Data-driven Approach to Use 1D Data for 3D Nonlinear Elastic Materials Modeling, *CMAME* (Submitted)

Regression

❖ Regression: prediction of response to an input based on *a priori* knowledge of the relationship between input and output data.

Input Data[1,2]

$\boldsymbol{\varepsilon}$	$\boldsymbol{\sigma}$
..	..
..	..



Find a relationship between stress and strain e.g. $\sigma = 2 * \varepsilon + 1$

1. Gather data from experiment or high-fidelity microstructure numerical simulation
2. Assume plain strain. $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ each contain three components

$\boldsymbol{\sigma} = f(\boldsymbol{\varepsilon})$ (hypothesis)

Hypothesis: linear or nonlinear relationship

Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

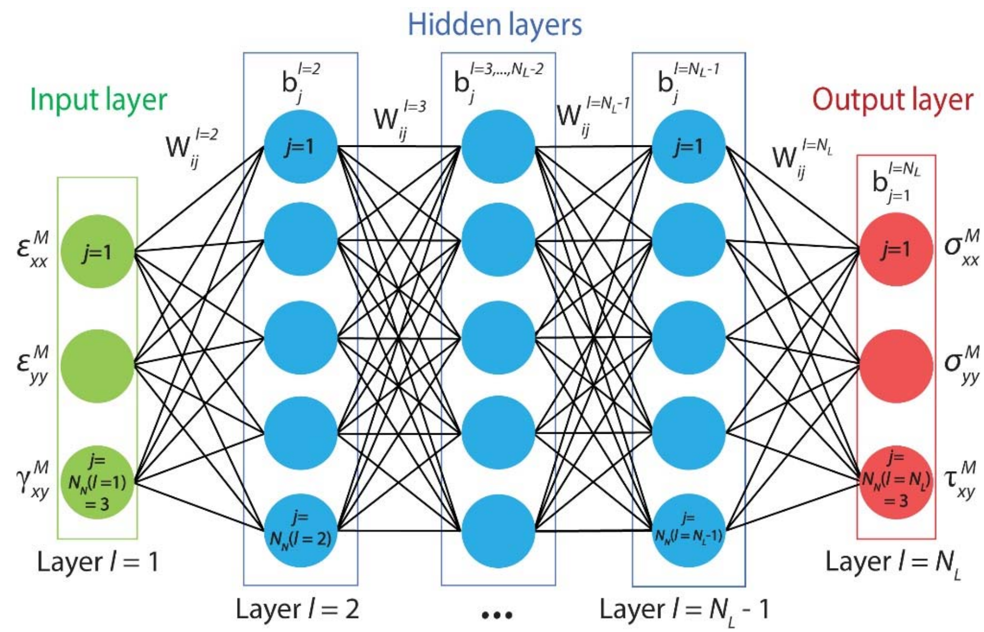
NORTHWESTERN UNIVERSITY

Regression using Feed Forward Neural Network (FFNN)

- The relationship between input and output can be explored using FFNN (an ANN with multiple hidden layers)

Input Data[1,2]

$\boldsymbol{\varepsilon}$	$\boldsymbol{\sigma}$
..	..
..	..



FFNN training: solving the following optimization problem

$$find: W_{ij}^{l=2},\ b_j^{l=2},\ W_{jk}^{l=3},\ b_k^{l=3}$$

$$min\ loss\ function: MSE = \frac{1}{N_T \times N_N(l=3)} \sum_{s=1}^{N_T} \sum_{k=1}^{N_N(l=3)} \left(\sigma_k^{l=3,s} - \sigma_k^{*,l=3,s}\right)^2$$

$$where: \sigma_k^{l=3,s} = \sum_{j=1}^{N_N(l=2)} W_{jk}^{l=3} \left(\mathscr{A}\left(\sum_{i=1}^{N_N(l=1)} W_{ij}^{l=2} \varepsilon_i^{M,s} + b_j^{l=2} \right) + b_k^{l=3} \right)$$

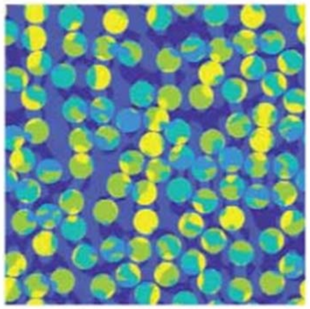
Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

NORTHWESTERN UNIVERSITY

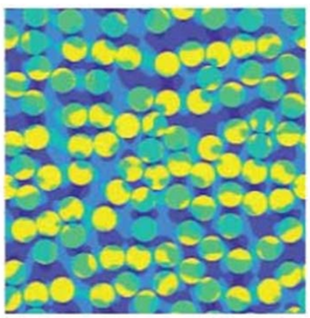
Regression using Convolutional Neural Network (CNN) for inverse modeling

For a microstructure with given **micro-stress** distribution, can CNN predict the **macro-strain**?

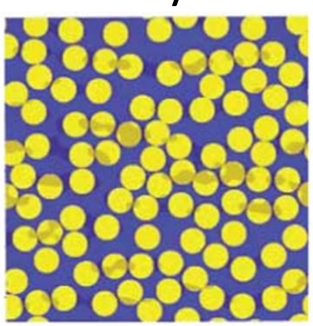
Micro-stress xx distribution



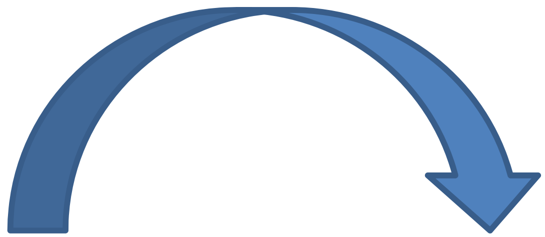
Micro−stress yy distribution



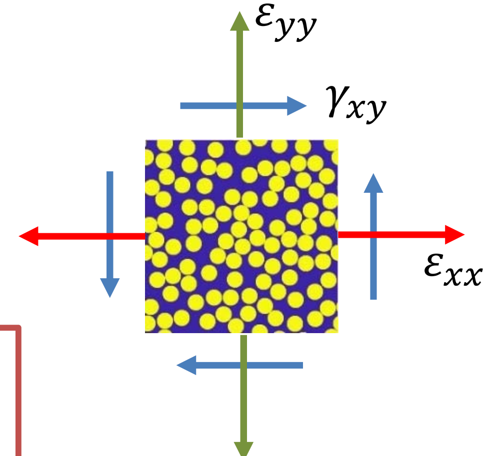
Micro-stress xy distribution



Inverse modeling



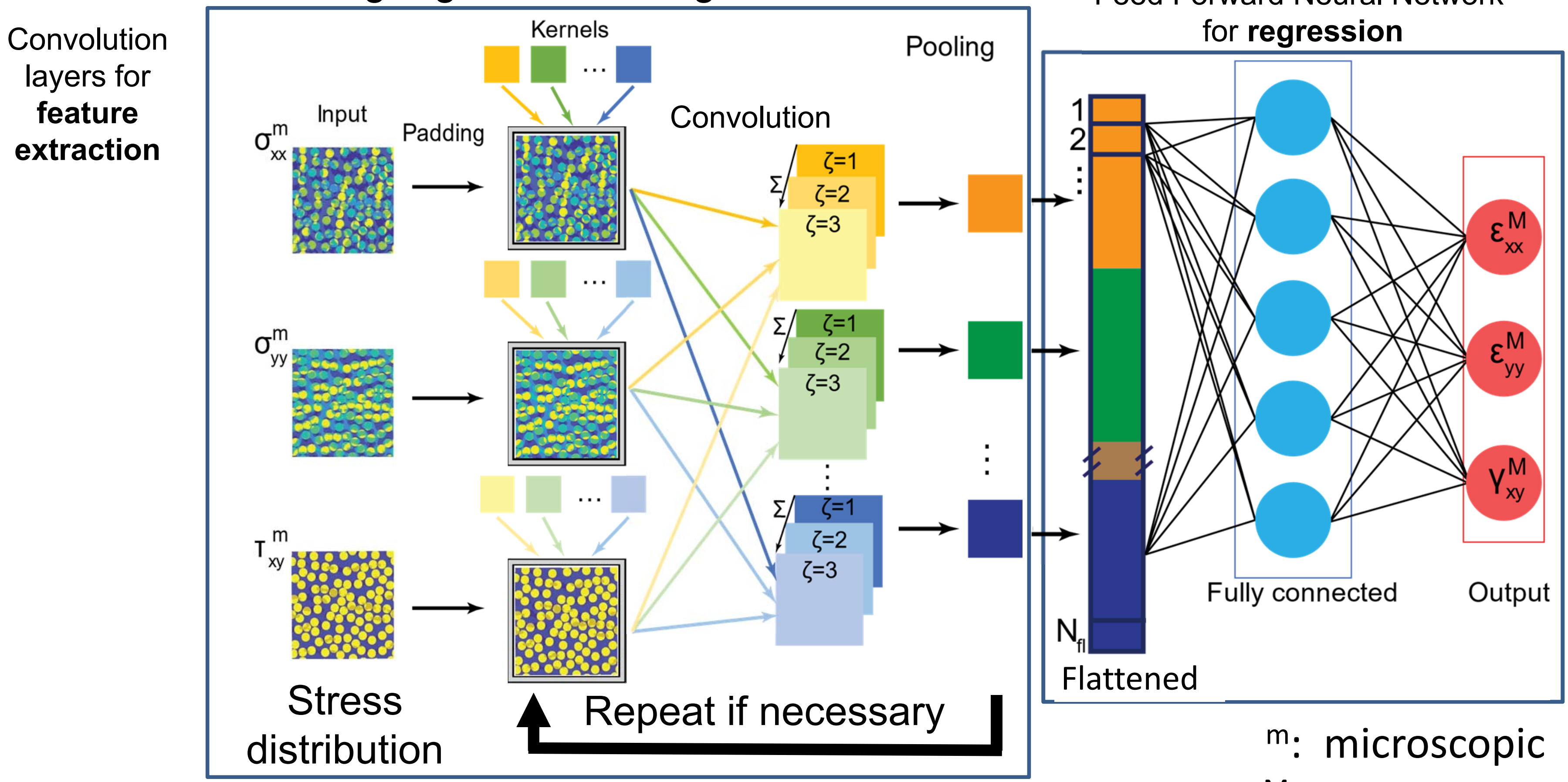
What is the macro-strain $\boldsymbol{\varepsilon}$?



Unidirectional microstructure

Regression using CNN for inverse modeling

Inverse modeling approach to obtain **macroscopic strain** from **microscopic stress distribution** using regression through CNN



^{m}: microscopic
^{M}: macroscopic

CNN training: solving the following optimization problem

$$find: W^l_{mn},\ b^l_n\ (l=2,3...N_L),\ \text{in FFNN}$$

$$W^{\zeta,\kappa,\eta}_{\xi,\psi}, b^{\kappa,\eta}(\kappa=1,2...N_{kernel}), (\eta=1,2...N_{conv}),\ \text{in Convolution layers}$$

$$min\ loss\ function: MSE = \frac{1}{N_T}\sum_{s=1}^{N_T}\left(\varepsilon^{M,s}-\varepsilon^{*M,s}\right)^2$$

$$where: \varepsilon^{M,s} = \mathscr{F}_{FFNN}\left(\mathscr{F}_{flatten}\left(\mathscr{F}^{N_{conv}}_{pcp}\left(...\mathscr{F}^2_{pcp}\left(\mathscr{F}^1_{pcp}\left(\sigma^s(\alpha,\beta)\right)\right)...\right)\right)\right)$$

Classification of damage

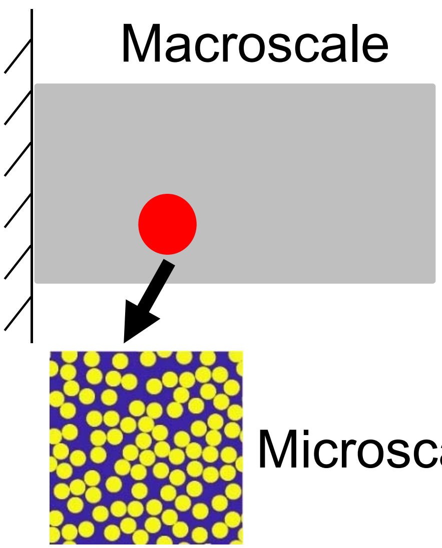
❖ Classification is a process of predicting the known class of given data points.

E.g., classify the state of the microstructure as “no damage” or “damage” based on local stress distribution

- Microscale material point damage is defined as: for any material point in the **microscale** domain, if the **micro-stress** exceeds certain threshold, the **micro** material point is damaged
- **Macroscale material point** damage is defined as:
  1) $P_D > P_{ND}$, damage in the microstructure
  2) $P_D < P_{ND}$, no damage in the microstructure

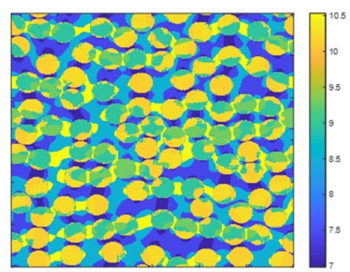
*P is probability

Macroscale

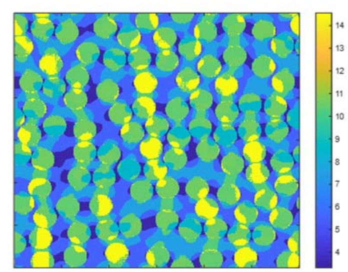


Microscale

Microscale



No damage



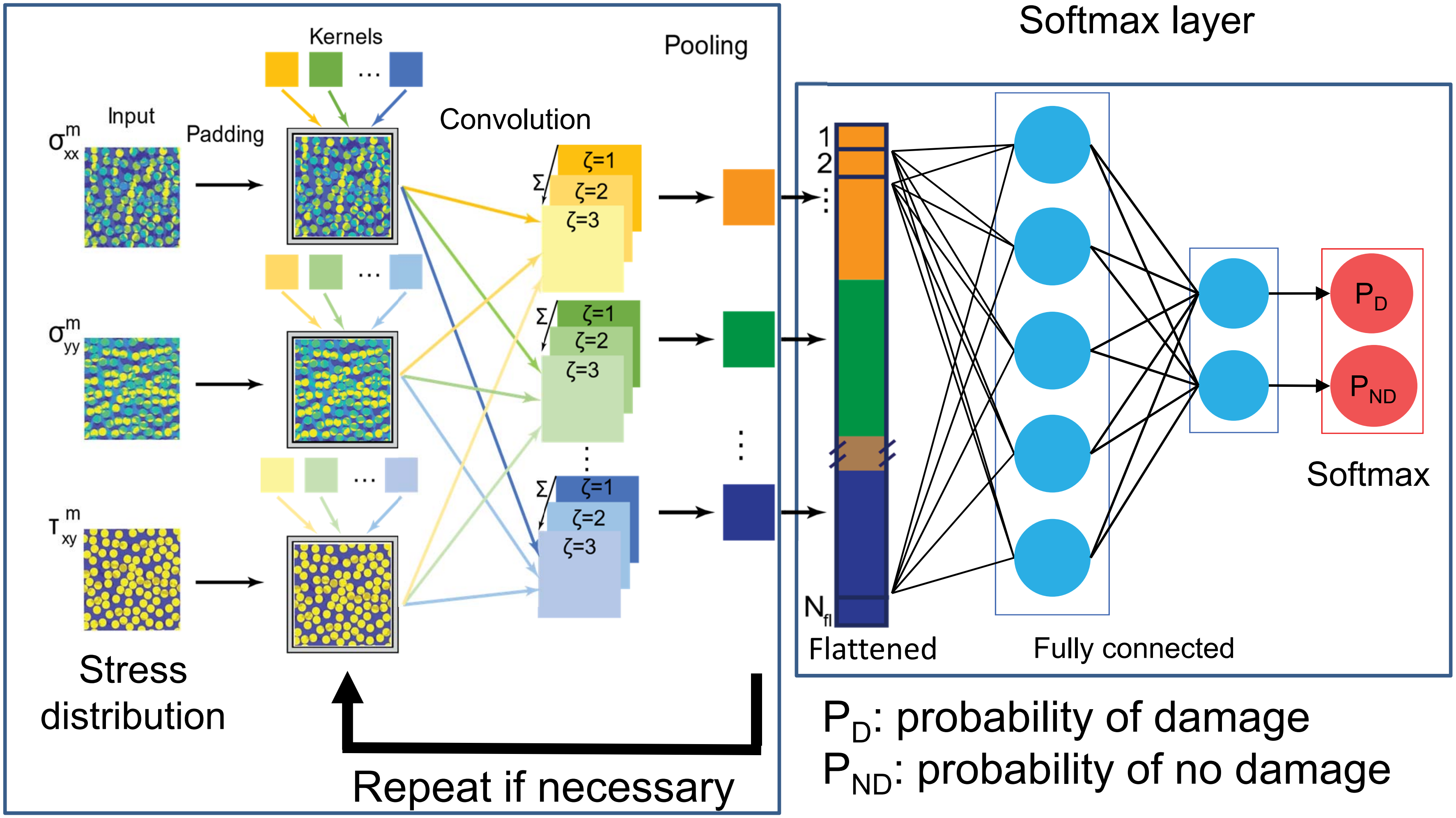
Damage

Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

Application of CNN for damage classification

Convolution layers for **feature extraction**

Feedforward Neural Network with Softmax layer



P_D: probability of damage

P_{ND}: probability of no damage

Macroscale material point damage is defined:

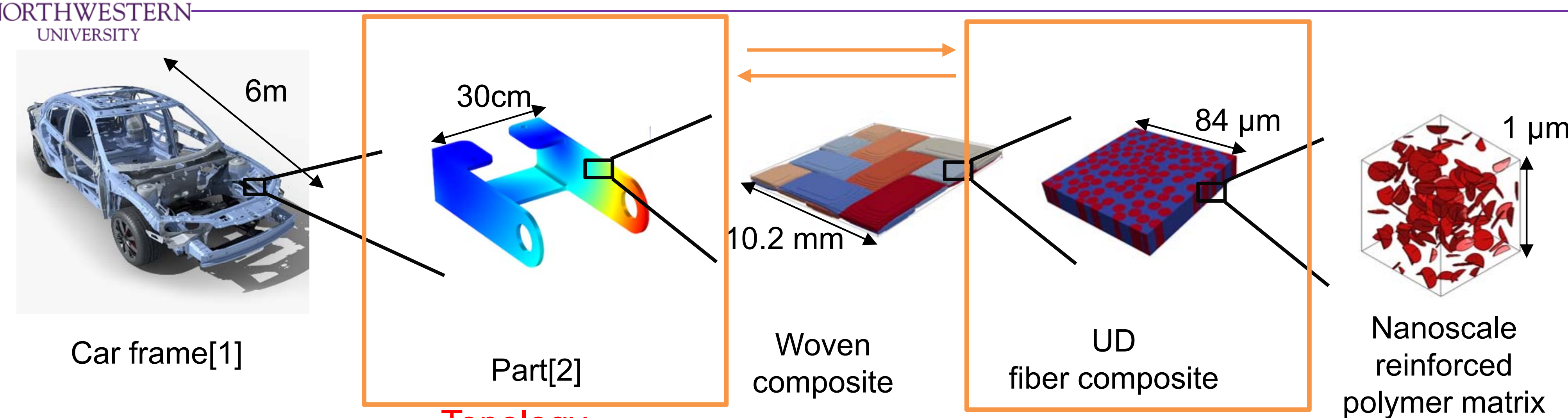
1) $P_D > P_{ND}$, damage in the microstructure
2) $P_D < P_{ND}$, no damage in the microstructure

Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

Outline

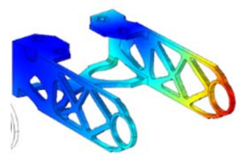
1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

Multiple length scales composite systems design & Optimization



Multiscale Design:

Topology



Density ditribution: $\rho(x)$

Woven thickness: h_{layer}

Layer angle: α_{layer}

Fiber radius: r_{fiber}

Fiber volume fraction: V_{fiber}

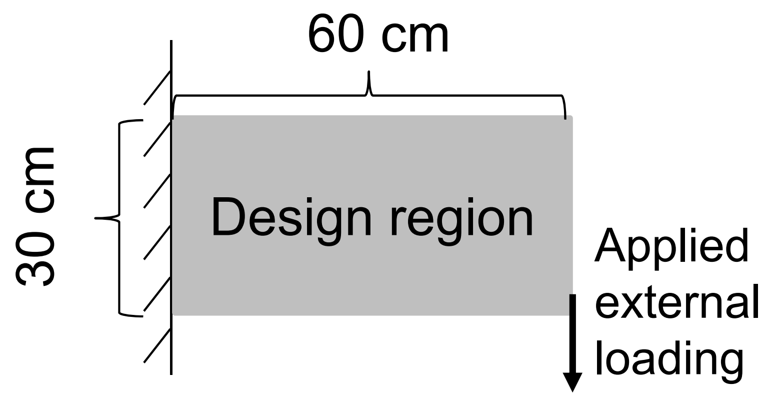
MoS_2 *volume fraction:* V_{Mos2}

Name	Part	Woven composite	UD composite	MoS2 polymer	Total
Length scale	cm	mm	μm	μm	-
Number of elements	10,000	40,000	360.000	90,000	1.296×10^{19}
Approximate optimization iterations	200	200	200	100	-
Total calculation cost	-	-	-	-	A tremendous number

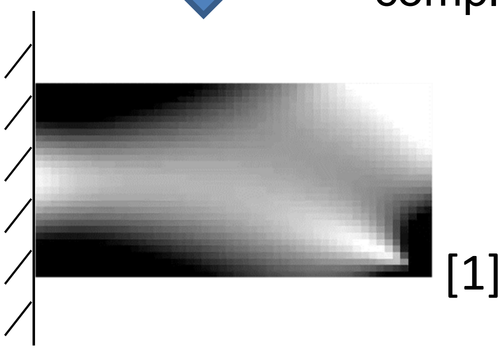
[1]https://www.cgtrader.com/3d-models/vehicle/part/car-frame-03
[2]https://www.comsol.com/blogs/performing-topology-optimization-with-the-density-method/

Topology optimization (TopOpt)

Single-scale topology optimization



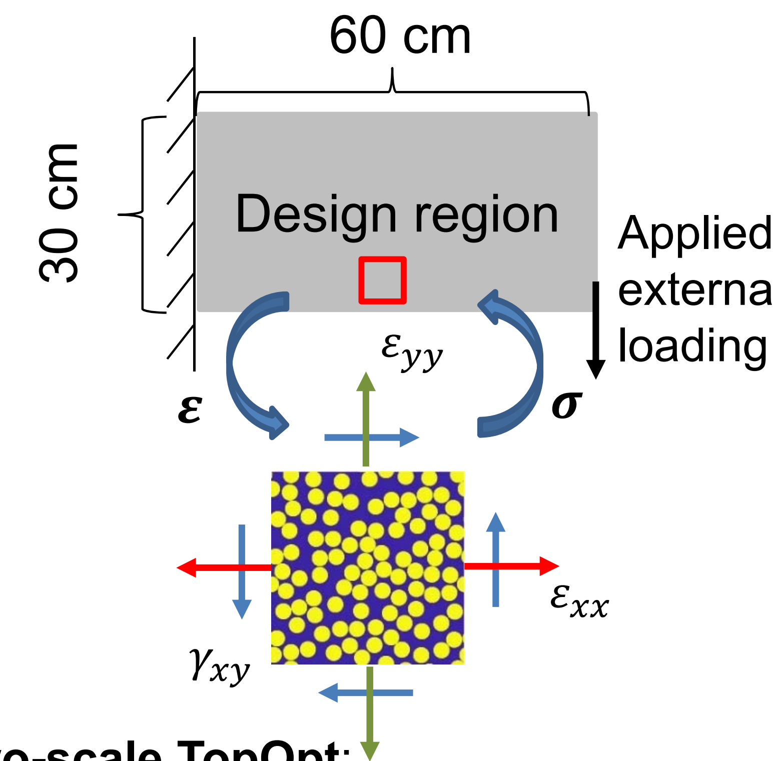
Minimizing system compliance



- Homogenous material assumed
- No microstructure
- Only elastic responses considered

[1] Sigmund, O. (2001). A 99 line topology optimization code written in Matlab. *Structural and multidisciplinary optimization*, *21*(2), 120-127.

Microstructure-based topology optimization is a two-scale problem

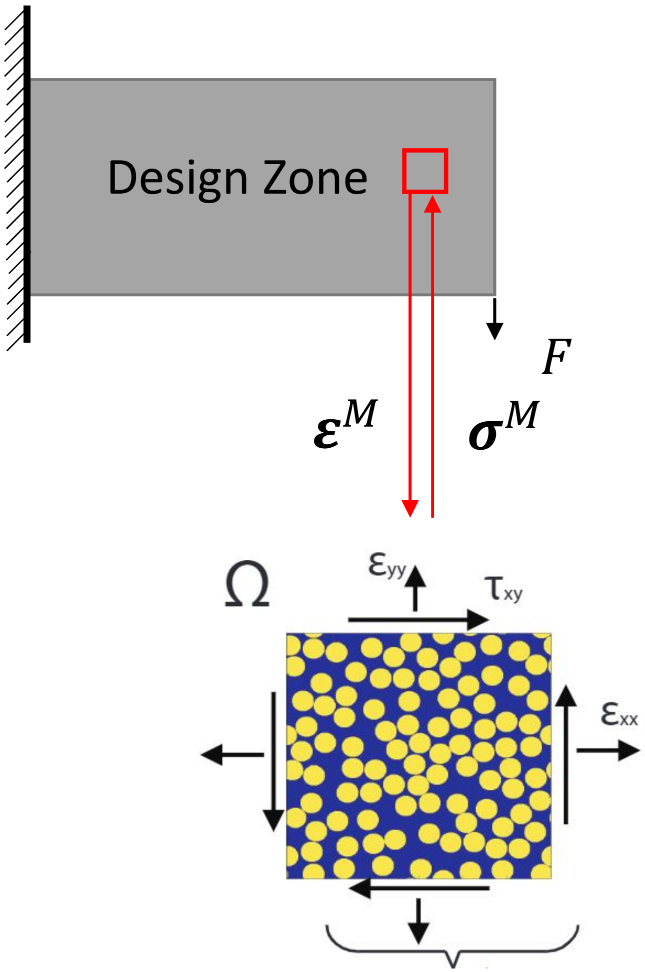


Two-scale TopOpt:
- Microstructures in all material points
- Design of microstructures and structure topology

- Evaluation of microstructure is time consuming during design iterations

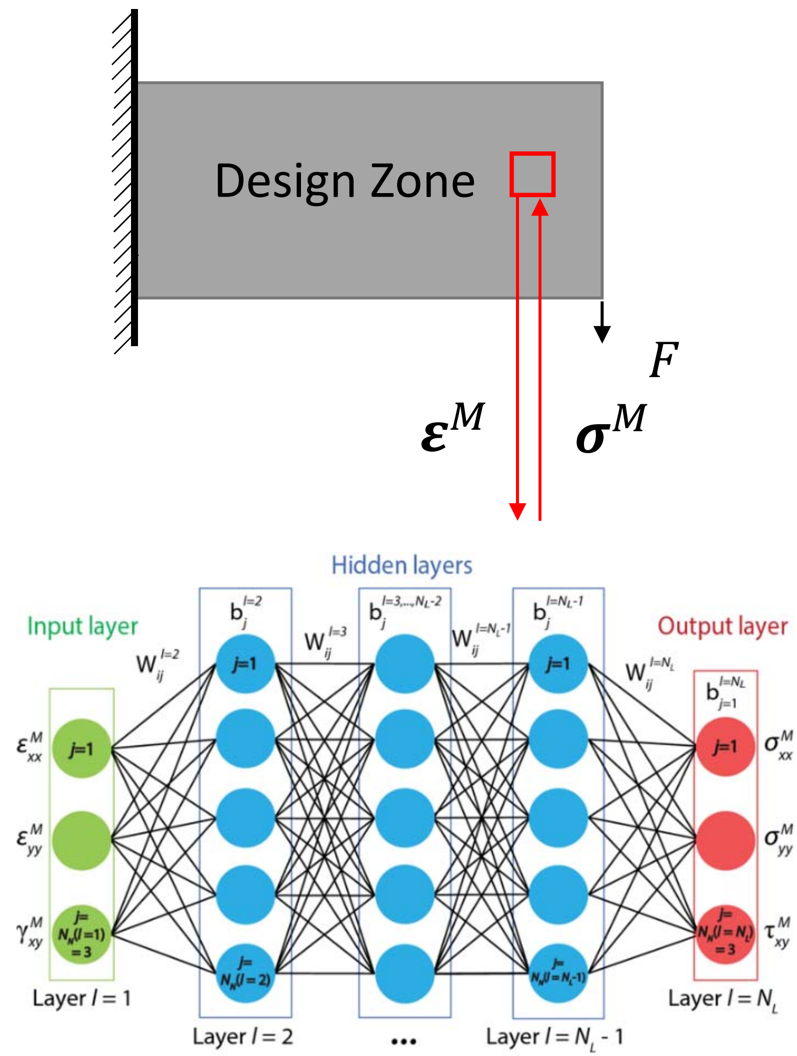
- Can FFNN and CNN approximate microstructure responses efficiently and accurately?

Topology optimization with FFNN



Representative Volume Element(RVE)

Training



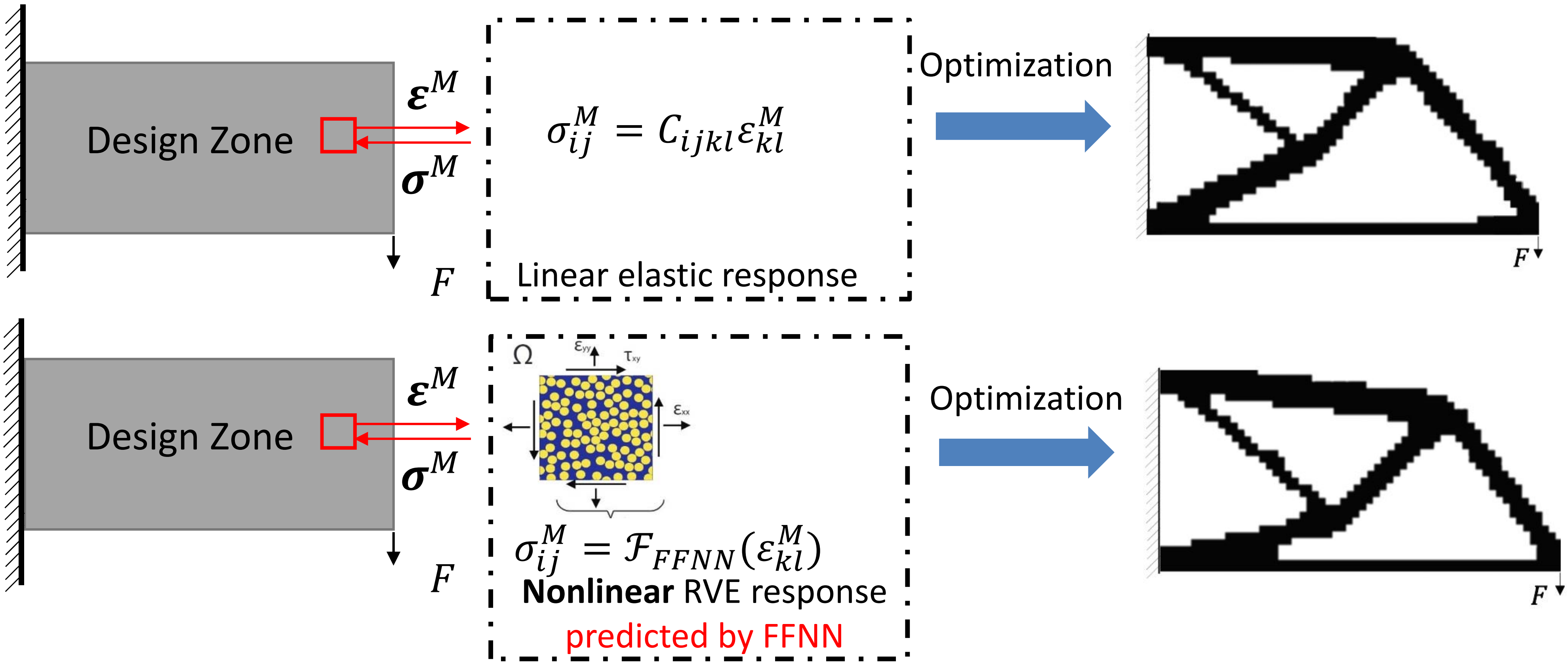
Feedforward Neural Network

FFNN approximates microstructure responses almost instantaneously

To be presented by Hengyang Li, 7/29/2019, 4:50-5:10pm, Room 202

Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

TopOpt with FFNN for nonlinear elastic materials

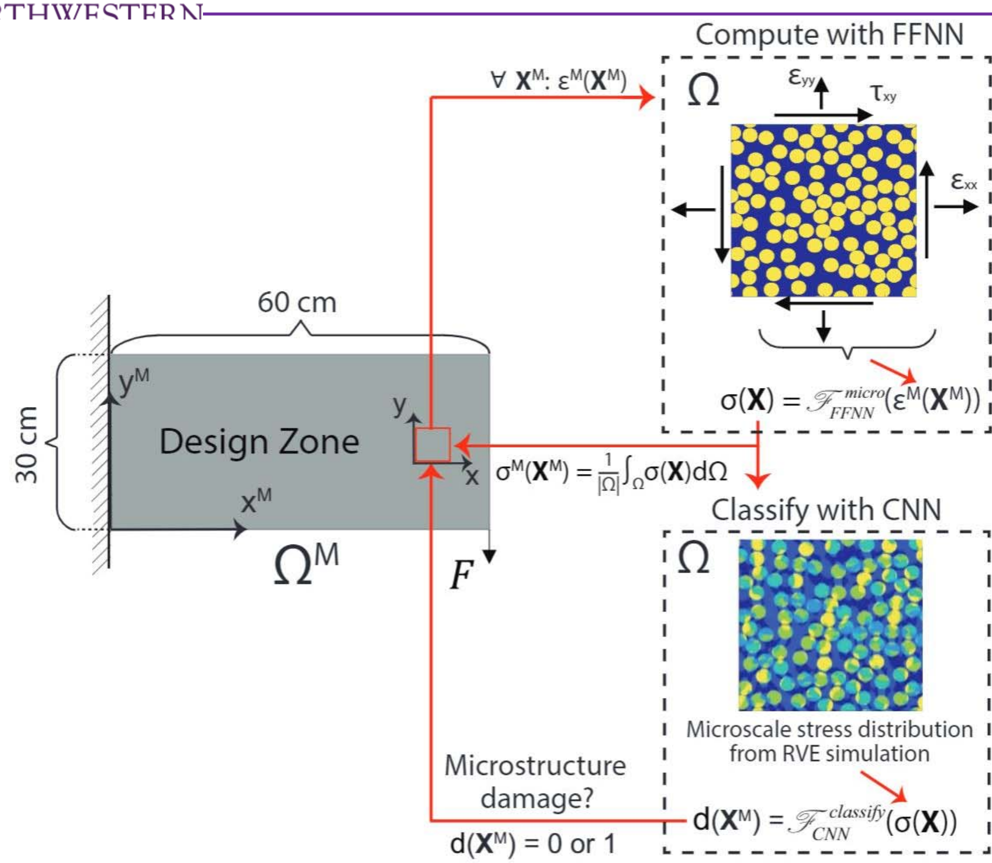


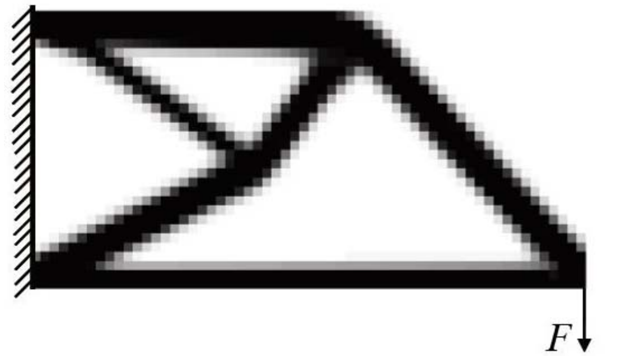
	Linear material	Nonlinear FEM-FEM two scale	Nonlinear FEM-FFNN two scale
Initial compliance $N \cdot cm$	295.0	-	375.0
Optimized compliance $N \cdot cm$	28.0	-	38.0
Optimization calculation time s	**338**	220×10^6	**472**
Factor of speed-up over FE-FE	-	-	**280,255**

Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

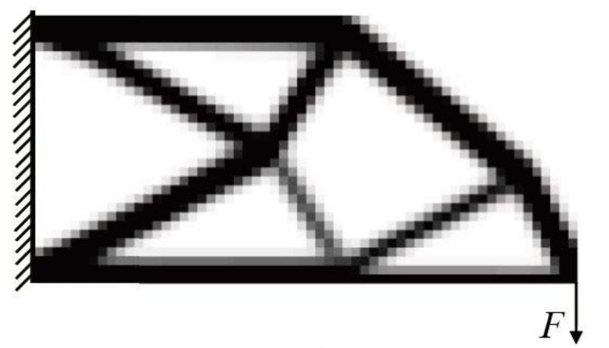
FEM: finite element method

TopOpt with FFNN+CNN for nonlinear elastic materials with damage constraints





Optimization with linear elastic material



Optimization with FFNN+CNN

	Linear material	Nonlinear FEM-FEM two scale	Nonlinear FEM-(FFNN+CNN) two scale
Initial compliance $N \cdot cm$	295.0	-	295
Optimized compliance $N \cdot cm$	30	-	31
Optimization calculation time s	**12.6**	$\mathbf{660 \times 10^6}$	**14.5**
Factor of speed-up over FE-FE	-	-	$\mathbf{45 \times 10^6}$

FEM: finite element method

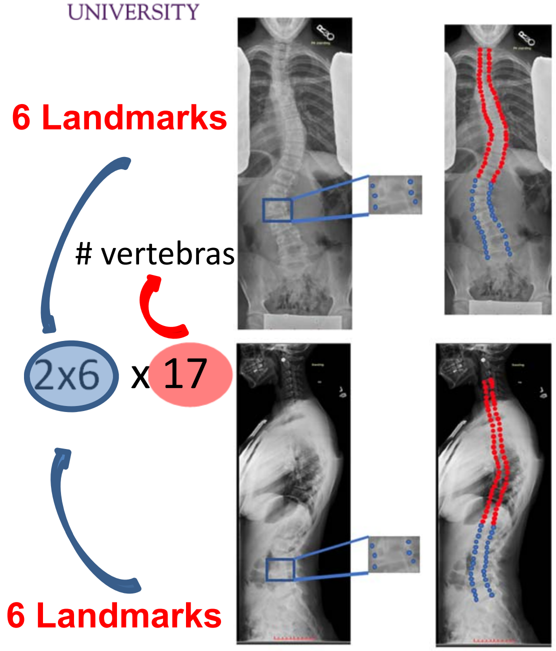
Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, Clustering discretization methods for generation of material performance databases in machine learning and design optimization, *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

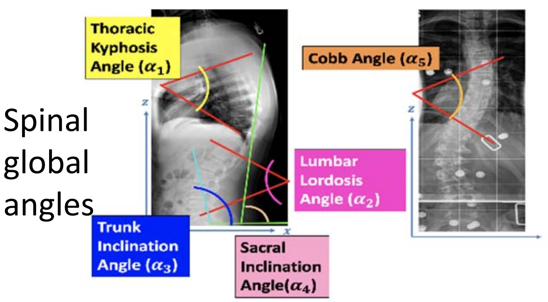
Outline

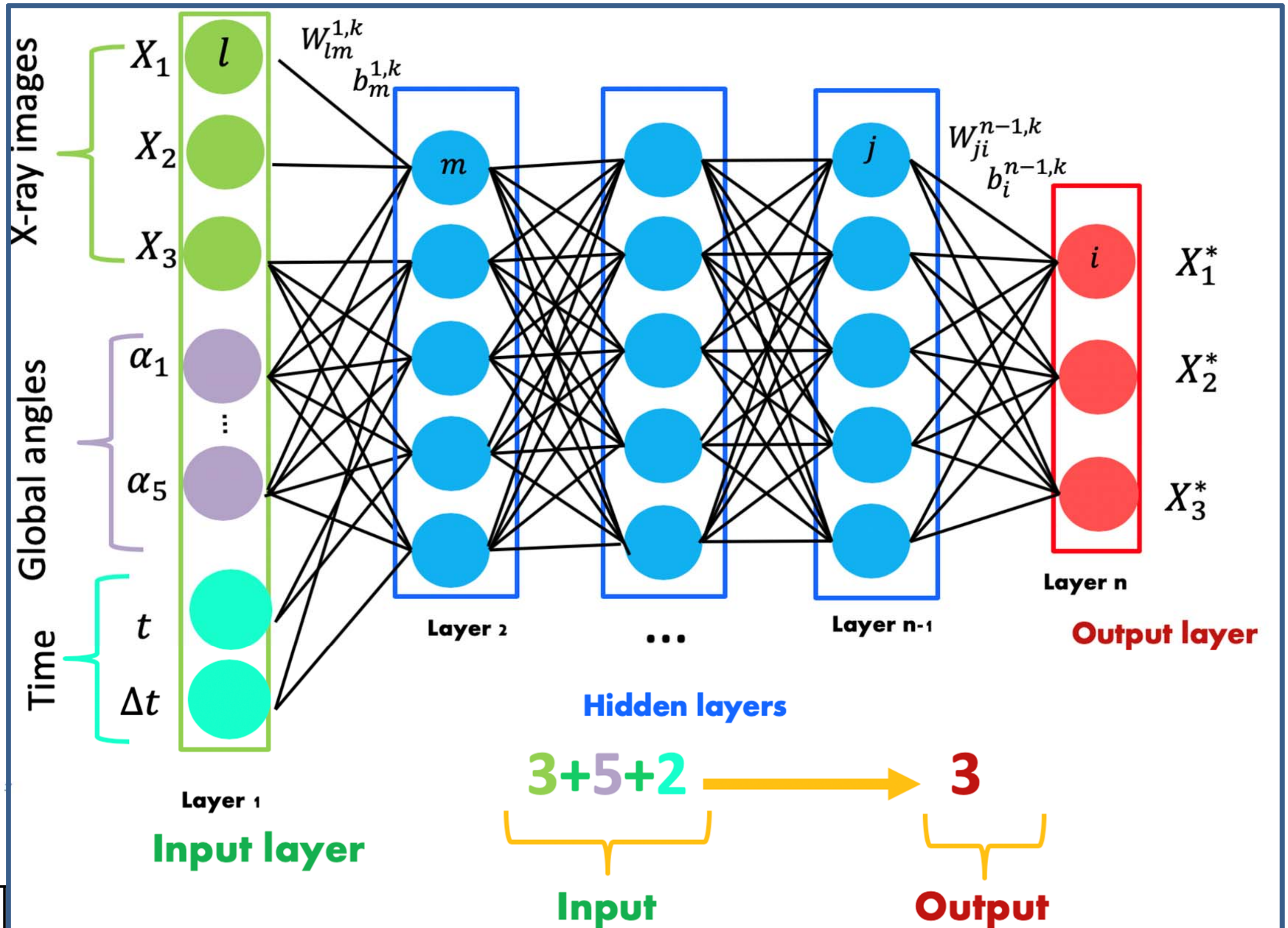
1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

Data-driven approach in predicting **A**dolescent **I**diopathic **S**coliosis (AIS)

NORTHWESTERN UNIVERSITY







Features	Data points (I)					
	1	2	3	.	.	Ns
X						
α	.	.	.	.	.	.
t	.	.	.	.	.	.
Δt						
X*						

X = Vector of input coordinates of a landmark $[X_1 \;\; X_2 \;\; X_3\;]$
α = Global angle vector $[\alpha_1\; \alpha_2\; \alpha_3\; \alpha_4\; \alpha_5\,]$
t = Age of the patient.
Δt = age variance between target age and current age (month).
X* = Vector of output co-ordinates of a landmark $[X_1^*\; X_2^*\; X_3^*]$.
Ns = Total number of landmarks = **2x6x17 = 204**

Using all features to train the Neural Network for all I=1,..,Ns

To be presented by Mahsa Tajdari, 7/31/2019, 2:40-3:00pm, Room 208

Physics Guided Neural Network (PGNN) to predict patient specific constants

Assume <u>linear</u> relationship between effective stress and **growth rate** $(\dot{X}_{mn})$ between time mΔt and nΔt

$$\dot{X}_{mn} = (\mathrm{m}-\mathrm{n})[\mathrm{A}(1+\mathrm{B}\times\sigma_{eff}^{n})]$$

Δt : unit of time (month)

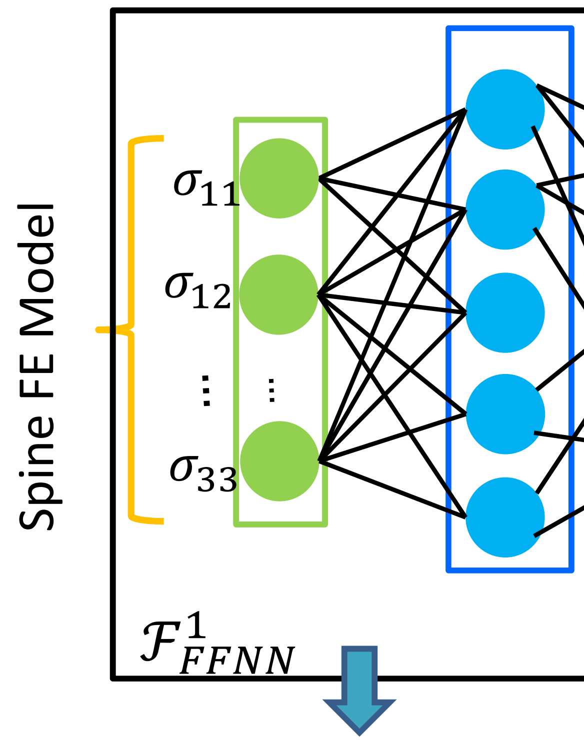
σ_{eff}^{n} : effective stress at time nΔt

A $(month^{-1})$and ***B***(MPa^{-1}) are patient specific constants that are calibrated inside the NN

Calculating Hyper Parameters in NN

X-ray 2D images

Generate patient specific model



Mean Square Error (MSE) corresponding $\mathcal{F}_{FFNN}^{1}$ $\longrightarrow$ $MSE_{X^{*,1}} = \mathrm{MSE}_1^1 + MSE_2^1 + MSE_3^1$

$$\mathrm{MSE}_1^1 = \mathrm{MSE}(\boldsymbol{X}^{*,1} - \widetilde{\boldsymbol{X}}^{*,1})$$

$$\mathrm{MSE}_2^1 = \mathrm{MSE}(\dot{X}_{mn} - \left((\mathrm{m}-\mathrm{n})\left[\boldsymbol{A_{TH}}(1+\boldsymbol{B_{TH}}\times\sigma_{eff\ (TH)}^{n})\right]\right))$$

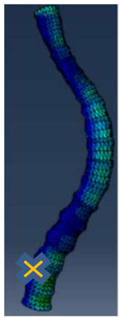
$$\mathrm{MSE}_3^1 = \mathrm{MSE}(\dot{X}_{mn} - \left((\mathrm{m}-\mathrm{n})\left[\boldsymbol{A}_{\mathrm{LU}}\left(1+\boldsymbol{B}_{\mathrm{LU}}\times\sigma_{eff\,(LU)}^{n}\right)\right]\right))$$

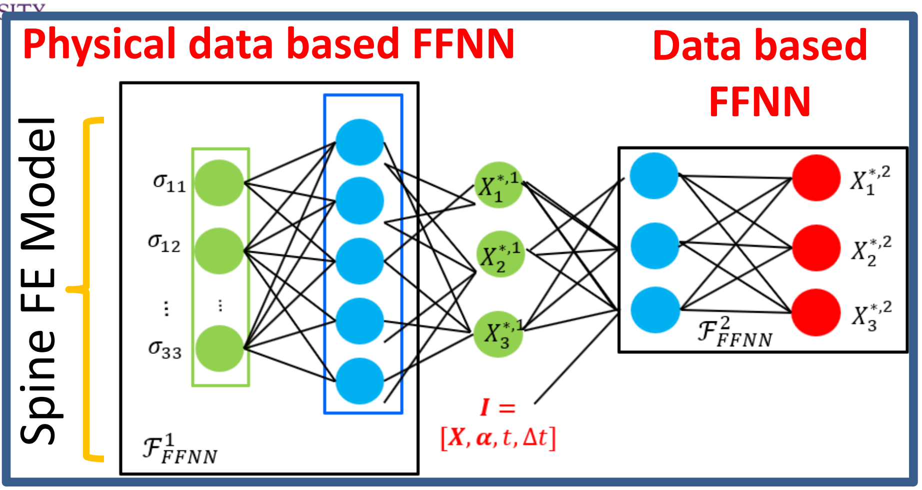
Hyper parameters ($\boldsymbol{A_{TH}}$, $\boldsymbol{B_{TH}}$, $\boldsymbol{A}_{\mathrm{LU}}$, $\boldsymbol{B}_{\mathrm{LU}}$) will be calculate inside NN

$\boldsymbol{A_{TH}}$ $(month^{-1})$ and $\boldsymbol{B_{TH}}(MPa^{-1})$ patient specific constants for Thoracic vertebrae

$\boldsymbol{A_{LU}}$ $(month^{-1})$ and $\boldsymbol{B_{LU}}(MPa^{-1})$ patient specific constants for Lumbar vertebrae

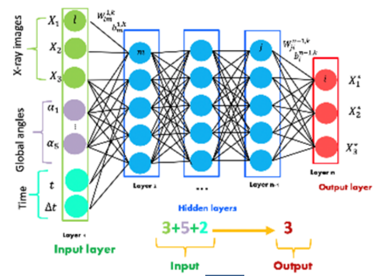
Composite NN using Multi-fidelity data



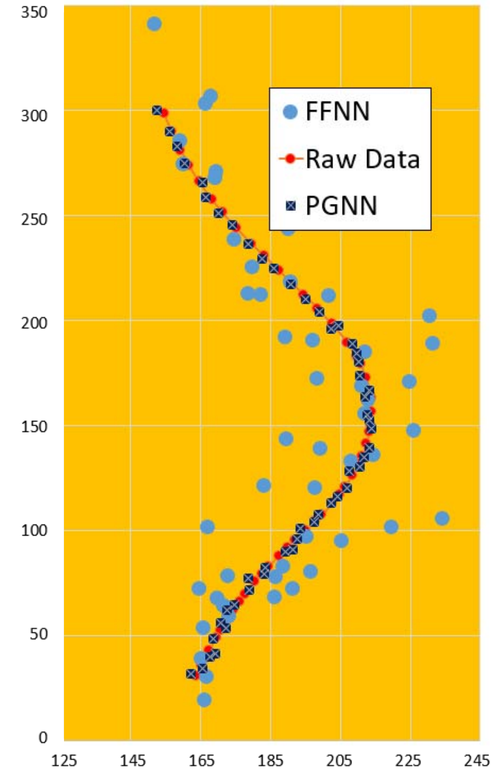


Raw Data

FFNN



PGNN



Physical Equation

$$\left((\mathrm{m}-\mathrm{n})\left[A_{TH}\left(1+B_{TH}\times\sigma^{n}_{eff\ (TH)}\right)\right]\right)-\dot{X}_{mn}=0$$

$$\left((\mathrm{m}-\mathrm{n})\left[A_{\mathrm{LU}}\left(1+B_{\mathrm{LU}}\times\sigma^{n}_{eff\,(LU)}\right)\right]\right)-\dot{X}_{mn}=0$$

α = Global angle vector $[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ ]$

t = Age of the patient.

Δt = age variance between target age and current age (month).

X* = Vector of output co-ordinates of a landmark $[X_1^*\ X_2^*\ X_3^*]$

X^m: position of data point in m^{th} month

X^n : position of data point in n^{th} month

$$\dot{X}_{mn}=\frac{\sqrt{(X_1^m-X_1^n)^2+(X_2^m-X_2^n)^2+(X_3^m-X_3^n)^2}}{\left(\sqrt{(X_1^n)^2+(X_2^n)^2+(X_3^n)^2}\right)}$$

Relative Error

Data based FFNN: 18.5%

PGNN: 4.63%

Outline

1. Motivation: source of data in mechanical science and engineering
2. Mechanistic Machine Learning (MML) for mechanical science and engineering
   - Interpretation of the data
   - Relevant concepts in data science
   - Introduction to different Machine Learning (ML) methods
     a. Unsupervised learning
     b. Supervised learning
3. Applications of ML methods
   1. Topology optimization
      1. Feed Forward Neural Network (FFNN)
      2. FFNN+ Convolutional Neural Network (CNN)
   2. Adolescent Idiopathic Scoliosis
      1. FFNN
      2. Physics Guided Neural Network (PGNN)
4. Why we need reduced order models/methods (ROM)
5. Summary and conclusions
6. References

Rich database of mechanical response information are necessary for training various Neural Networks

- Multiscale design and optimization is not feasible with direct microstructure responses calculation with Finite Element Method (FEM)
- Well-trained **NNs accelerates** microstructure and structure **design process**, e.g. Topology Optimization
- **Material microstructure responses database** is required for the training process.
- The database includes:
  - Macro-strain and macro-stress pairs
  - Micro-stress distribution and macro-strain pairs
  - Other microstructure quantities of interest

The gap:

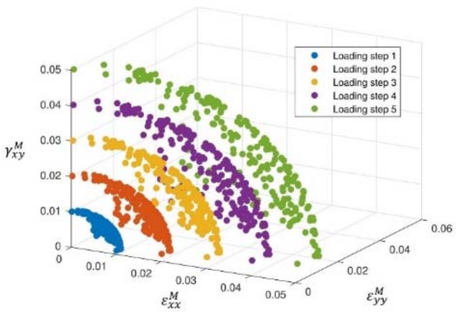
- ❑ Microstructure response simulation can be expensive using FEM
- ❑ Rich database requires a lot of runs of microstructure simulation

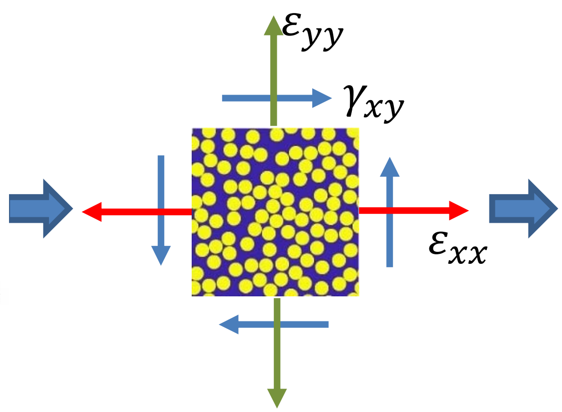
NORTHWESTERN UNIVERSITY

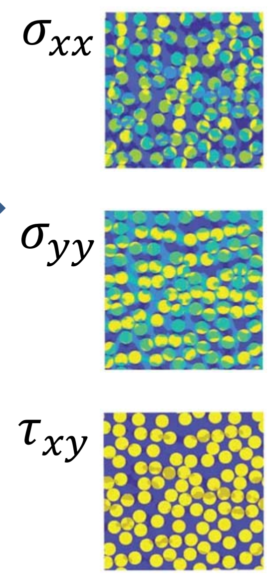
Cost of microstructure responses database generation

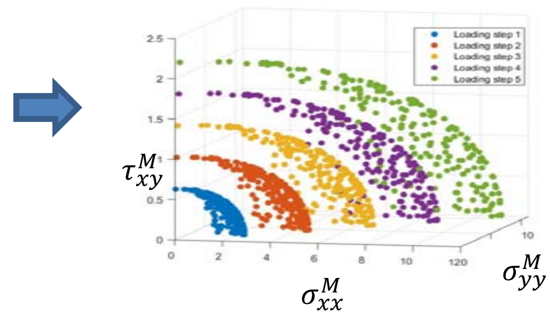
1,000 load cases for training a 2D hyper elastic problem:

External loading states | Microscopic stress | Averaged stress









$$\boldsymbol{\varepsilon}^n = [\varepsilon_{xx}^n, \varepsilon_{yy}^n, \gamma_{xy}^n]$$
$$n = 1,2 \dots, 1000$$

600 x 600 x 3 x 1000

$$\boldsymbol{\sigma}^n = [\sigma_{xx}^n, \sigma_{yy}^n, \tau_{xy}^n]$$
$$n = 1,2 \dots, 1000$$

Running 1,000 microstructure simulation is expensive:

Microstructure simulation method	Total simulation time (s)
FFT	3.01×10^5
FEM	2.04×10^7

HPC is needed

Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, 'Clustering discretization methods for generation of material performance databases in machine learning and design optimization', *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

Approximate material responses using FFNN

External loading states

Data point	Feature
1	$\boldsymbol{\varepsilon}^1$
2	$\boldsymbol{\varepsilon}^2$
…	
n	$\boldsymbol{\varepsilon}^n$

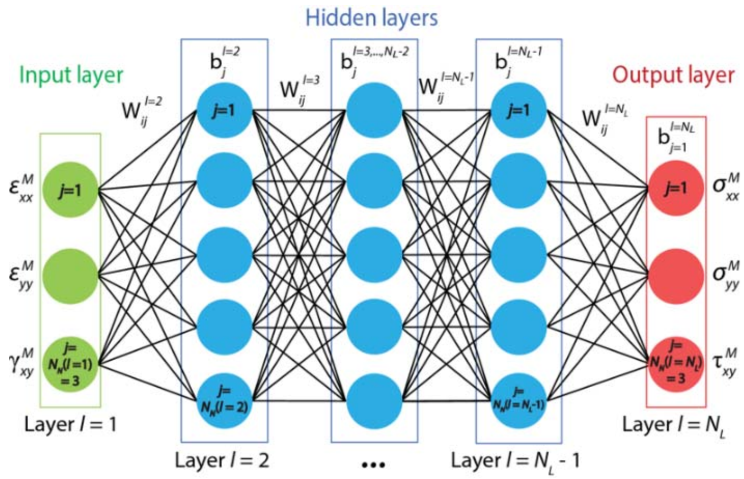
$$\boldsymbol{\varepsilon}^n = [\varepsilon_{xx}^n, \varepsilon_{yy}^n, \gamma_{xy}^n]$$
$$n = 1,2 \ldots, 1000$$

Averaged stress

Data point	Feature
1	$\boldsymbol{\sigma}^1$
2	$\boldsymbol{\sigma}^2$
…	
n	$\boldsymbol{\sigma}^n$

$$\boldsymbol{\sigma}^n = [\sigma_{xx}^n, \sigma_{yy}^n, \tau_{xy}^n]$$
$$n = 1,2 \ldots, 1000$$

FFNN with well-trained parameters



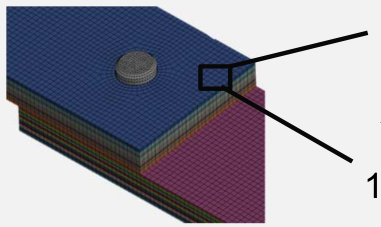
How to generate microstructure responses efficiently?

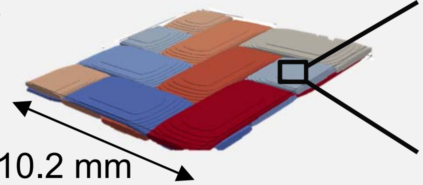
Li, H, Kafka, OL, Gao, J, Yu, C, Nie, Y, Zhang, L, Tajdari, M, Tang, S, Guo, X, Li, G, Tang, S, Cheng, G & Liu, WK 2019, 'Clustering discretization methods for generation of material performance databases in machine learning and design optimization', *Computational Mechanics*. https://doi.org/10.1007/s00466-019-01716-0

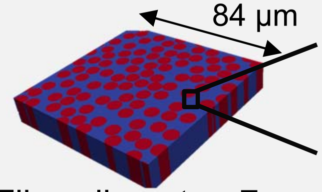
NORTHWESTERN UNIVERSITY

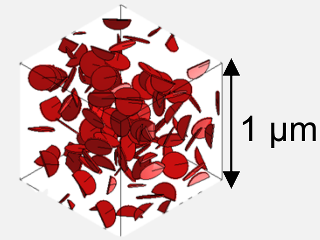
Curse of dimensionality in complex material systems

Material Scale:	Single lap bolt joint	Woven composite	UD composite	MoS_2 reinforced polymer
Typical number of elements:	1x10^6	2x10^6	36x10^6	64x10^3
Clusters used:	10 clusters	96 clusters	10 clusters	10 clusters









FEM

$$1.38\times10^{25}\left\{\underbrace{\begin{bmatrix}\{\mathbf{K}\}^{(1)(1)} & \{\mathbf{K}\}^{(1)(2)} & \cdots & \{\mathbf{K}\}^{(1)(4)} \\ \{\mathbf{K}\}^{(2)(1)} & \{\mathbf{K}\}^{(2)(2)} & \cdots & \{\mathbf{K}\}^{(2)(4)} \\ \vdots & \vdots & \ddots & \vdots \\ \{\mathbf{K}\}^{(4)(1)} & \{\mathbf{K}\}^{(4)(2)} & \cdots & \{\mathbf{K}\}^{(4)(4)}\end{bmatrix}}_{1.38\times10^{25}}\right.\begin{bmatrix}\{\delta\mathbf{d}\}^{(1)} \\ \{\delta\mathbf{d}\}^{(2)} \\ \vdots \\ \{\delta\mathbf{d}\}^{(4)}\end{bmatrix} = \begin{bmatrix}\{\mathbf{r}\}^{(1)} \\ \{\mathbf{r}\}^{(2)} \\ \vdots \\ \{\mathbf{r}\}^{(4)}\end{bmatrix}$$

Strong interaction between scales

DoFs using FEM:

$$3\left(N_n^{(1)} + N_i^{(1)}N_n^{(2)} + \cdots + N_i^{(1)}N_i^{(2)}N_i^{(3)}N_n^{(4)}\right)$$

Number of elements using **FEM**:

1x10^6 $*$ 2.6x10^6 $*$ 36x10^6 $*$ 64x10^3 = **6x10^{25} elements**

N_i: number of integration points
N_n: number of nodes
Superscripts indicate scale level

Extremely large problem

MCA

$$639420\left\{\underbrace{\begin{bmatrix}\{\mathbf{M}\}^{(1)(1)} & \{\mathbf{M}\}^{(1)(2)} & \cdots & \{\mathbf{M}\}^{(1)(4)} \\ \{\mathbf{M}\}^{(2)(1)} & \{\mathbf{M}\}^{(2)(2)} & \cdots & \{\mathbf{M}\}^{(2)(4)} \\ \vdots & \vdots & \ddots & \vdots \\ \{\mathbf{M}\}^{(4)(1)} & \{\mathbf{M}\}^{(4)(2)} & \cdots & \{\mathbf{M}\}^{(4)(4)}\end{bmatrix}}_{639420}\right.\begin{bmatrix}\{\delta\boldsymbol{\varepsilon}\}^{(1)} \\ \{\delta\boldsymbol{\varepsilon}\}^{(2)} \\ \vdots \\ \{\delta\boldsymbol{\varepsilon}\}^{(4)}\end{bmatrix} = \begin{bmatrix}\{\mathbf{r}\}^{(1)} \\ \{\mathbf{r}\}^{(2)} \\ \vdots \\ \{\mathbf{r}\}^{(4)}\end{bmatrix}$$

DoFs using MCA:

$$6\left(N_c^{(1)} + \cdots + N_c^{(1)}N_c^{(2)}N_c^{(3)}N_c^{(4)}\right)$$

N_c: number of clusters

$N_c \ll N_n$ at each scale

Number of clusters using **MCA**:

10 * 96 * 10 *10 = **9.6x10^4 clusters**

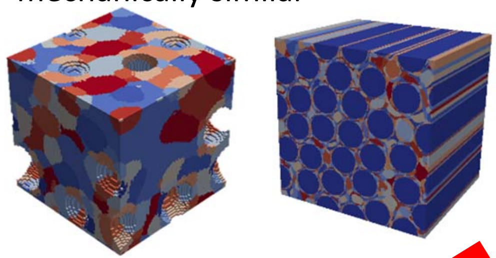
Solvable on small HPC/single PC

Two-scale theory: **S**elf-consistent **C**lustering **A**nalysis (SCA) overview

- ***Objective:*** Efficient and accurate homogenization of nonlinear history dependent heterogeneous materials with complex microstructure.

Data-driven order reduction

Group points in the MVE that are mechanically similar

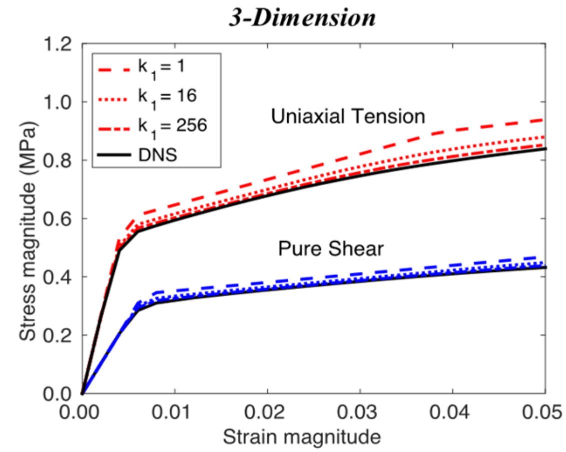


Mechanistic prediction

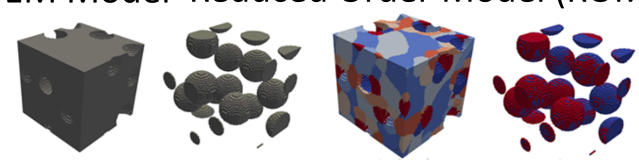
- Lippmann-Schwinger integral equation
- Micromechanics mean-field theory

Self-consistent Clustering Analysis (SCA)

Numerical verification



FEM Model Reduced Order Model (ROM)



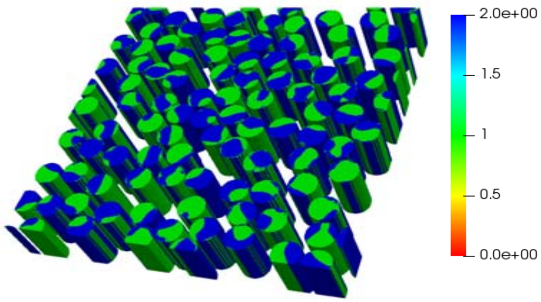
	System Complexity	Computational time	Speed-up
FEM	80x80x80	25.7 hr (24 cores)	1
ROM (SCA)	16	2 s	1×10^6
	256	50 s	5×10^4

1. Liu, Z., Bessa, M. A., & Liu, W. K. (2016). Computer Methods in Applied Mechanics and Engineering
2. Liu, Z., Fleming, M., & Liu, W. K. (2018). Computer Methods in Applied Mechanics and Engineering
3. Bessa, M. A., Bostanabad, R., Liu, Z., Hu, A., Apley, D. W., Brinson, C., ... & Liu, W. K. (2017). Computer Methods in Applied Mechanics and Engineering
4. Liu, Z., Kafka, O. L., Yu, C., & Liu, W. K. (2018). In Advances in Computational Plasticity
5. Tang, S., Zhang, L., & Liu, W. K. (2018). Computational Mechanics
6. Kafka, O. L., Yu, C., Shakoor, M., Liu, Z., Wagner, G. J., & Liu, W. K. (2018). JOM
7. Shakoor, M., Kafka, O. L., Yu, C., & Liu, W. K. (2018). Computational Mechanics
8. Li, H., Kafka, O. L., Gao, J., Yu, C., Nie, Y., Zhang, L., ... & Tang, S. (2019). Computational Mechanics
9. Zhang, L., Tang, S., Yu, C., Zhu, X., & Liu, W. K. (2019). Computational Mechanics
10. Gao, J., Shakoor, M., Jinnai, H., Kadowaki, H. Seta, E., Liu. W. K. An Inverse Modeling Approach for Predicting Filled Rubber Performance. (2019) Computer Methods in Applied Mechanics and Engineering

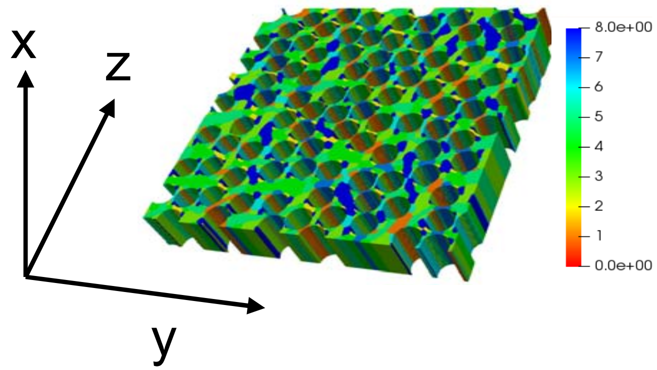
To be covered in Lecture 2

Why use reduced order modeling for data generation?

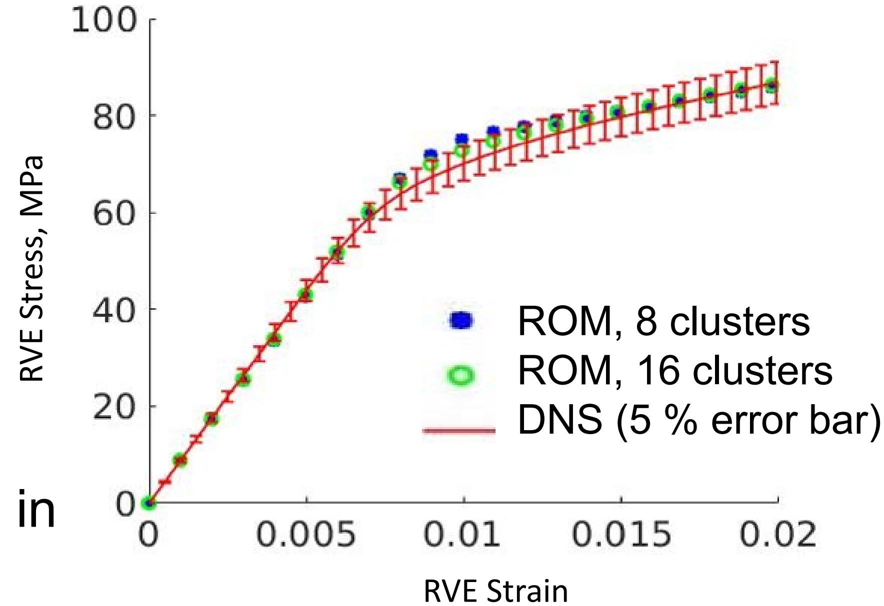
Fiber phase (2 clusters)



Matrix phase (8 clusters)



Apply loading in y direction



DNS: **~200 hr** using **80 cpus**

ROM: 2 s using 1 cpu

- Material design requires a large database of microstructure response information
- Reduced order modeling (ROM) allows fast data generation for:
  - Different heterogeneous microstructures
  - Different material constituents

The microstructure database generation can now be done on single PC

Conclusions

- Rich datasets provide us an opportunity to integrate mechanical and data sciences for rapid prediction, design, and optimization.
- Data science enables solution of large-scale problems, otherwise not tractable using current methodologies.
- Reduce Order Models (ROM) such as Principal Component Analysis (PCA), Self-consistent Clustering Analysis (SCA), Multiresolution Clustering Analysis (MCA), help us rapidly generate key datasets.
- Machine learning techniques such as neural networks (FFNN, CNN, PGNN, etc.) can augment ROMs for extremely fast computations.
- Combining ROMs with machine learning techniques has the potential to discover, design, and optimize novel complex material systems.
- Mathematical theories for biological systems are in their infancy; discovery of hypotheses in biological system might be achieved by considering physics, e.g. via a physic guided neural network

Acknowledgement

Our thanks to…

Funding agencies:

- Air Force Office of Scientific Research
- Army Research Office
- Beijing Institute of Collaborative Innovation
- Bridgestone Corporation
- DMDII (UI Labs)
- Ford Motor Company/National Energy Technology Laboratory
- Lurie Children's Hospital
- NIST CHiMaD I and CHiMaD II
- NIST Gaithersburg
- Northwestern University Data Science Initiative
- NSF
    - GRFP DGE-1324585
    - CMMI MOMS
    - CMMI CPS

NORTHWESTERN UNIVERSITY

References

[1] Li, H., Kafka, O. L., Gao, J., Yu, C., Nie, Y., Zhang, L., Tajdari, M., Tang, S., Guo, X., Li, G., Tang, S., Cheng, G., & Liu, W. K. (2019). Clustering discretization methods for generation of material performance databases in machine learning and design optimization. Computational Mechanics
[2] Boyd, S., & Vandenberghe, L. (2004). Convex optimization. Cambridge university press.
[3] Zhang, L., Yang, Y., Li H., Gao J., Reno D., Tang S., Liu W.K. Neural network finite element method, in preparation
[4] Approximation by superpositions of a sigmoidal function, by George Cybenko (1989)
[5] Multilayer feedforward networks are universal approximators, by Kurt Hornik, Maxwell Stinchcombe, and Halbert White (1989).
[6] Zhang, L., Yang, Y., Li H., Gao J., Reno D., Tang S., Liu W.K. Neural network finite element method, in preparation
[7] Watt, J., Borhani, R., & Katsaggelos, A. K. (2016).Machine learning refined: foundations, algorithms, and applications. Cambridge University Press.
[8] Tang, S., Zhang, G., Yang, H., Guo, X., Li, Y., & Liu, W. K. (2019). MAP123: A Data-driven Approach to Use 1D Data for 3D Nonlinear Elastic Materials Modeling, Computer Methods in Applied Mechanics and Engineering (Submitted)
[9] Sigmund, O. A 99 line topology optimization code written in Matlab. (2001). Structural and multidisciplinary optimization
[10] Liu, Z., Bessa, M. A., & Liu, W. K. (2016). Computer Methods in Applied Mechanics and Engineering
[11] Liu, Z., Fleming, M., & Liu, W. K. (2018). Computer Methods in Applied Mechanics and Engineering
[12] Bessa, M. A., Bostanabad, R., Liu, Z., Hu, A., Apley, D. W., Brinson, C., ... & Liu, W. K. (2017). Computer Methods in Applied Mechanics and Engineering
[13] Liu, Z., Kafka, O. L., Yu, C., & Liu, W. K. (2018). In Advances in Computational Plasticity
[14] Tang, S., Zhang, L., & Liu, W. K. (2018). Computational Mechanics
[15] Kafka, O. L., Yu, C., Shakoor, M., Liu, Z., Wagner, G. J., & Liu, W. K. (2018). JOM
[16] Shakoor, M., Kafka, O. L., Yu, C., & Liu, W. K. (2018). Computational Mechanics
[17] Li, H., Kafka, O. L., Gao, J., Yu, C., Nie, Y., Zhang, L., ... & Tang, S. (2019). Computational Mechanics
[18] Zhang, L., Tang, S., Yu, C., Zhu, X., & Liu, W. K. (2019). Computational Mechanics
[19] Gao, J., Shakoor, M., Jinnai, H., Kadowaki, H. Seta, E., Liu. W. K. An Inverse Modeling Approach for Predicting Filled Rubber Performance. (2019) Computer Methods in Applied Mechanics and Engineering